DOCUMENT RESUME

ED 160 452 SE 025 101

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TITLE Geometry with Coordinates, Student's Text, Part I, Unit 47. Revised Edition.
INSTITUTION Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 63
NOTE 514p.; For related documents, see SE 025 102-104

EDRS PRICE MF-$1.00 HC-$27.45 Plus Postage.
DESCRIPTORS *Analytic Geometry; Curriculum; *Geometry; *Instructional Materials; Mathematics Education; Secondary Education; *Secondary School Mathematics; *Textbooks
IDENTIFIERS *School Mathematics Study Group

ABSTRACT

This is part one of a two-part SMSG geometry text for high school students. One of the goals of the text is the development of analytic geometry hand-in-hand with synthetic geometry. The authors emphasize that both are deductive systems and that it is useful to have more than one mode of attack in solving problems. The text begins the development of geometry synthetically and teaches the method of synthetic proof, then leads quickly to the use of coordinate systems in the remainder of the work. Chapter topics include: introduction to formal geometry; sets, points, lines, and planes; distance and coordinate systems; angles; congruence; parallelism; and similarity. (MN)

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SE 025 101

Geometry with Coordinates

Student's Text, Part I

REVISED EDITION

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New Haven and London, Yale University Press, 1963

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Financial support for the School Mathematics
Study Group has been provided by the
National Science Foundation.

FOREWORD

The increasing contribution of mathematics to the culture of the modern world, as well as its importance as a vital part of scientific and humanistic education, has made it essential that the mathematics in our schools be both well selected and well taught.

With this in mind, the various mathematical organizations in the United States cooperated in the formation of the School Mathematics Study Group (SMSG). SMSG includes college and university mathematicians, teachers of mathematics at all levels, experts in education, and representatives of science and technology. The general objective of SMSG is the improvement of the teaching of mathematics in the schools of this country. The National Science Foundation has provided substantial funds for the support of this endeavor.

One of the prerequisites for the improvement of the teaching of mathematics in our schools is an improved curriculum--one which takes account of the increasing use of mathematics in science and technology and in other areas of knowledge and at the same time one which reflects recent advances in mathematics itself. One of the first projects undertaken by SMSG was to enlist a group of outstanding mathematicians and mathematics teachers to prepare a series of textbooks which would illustrate such an improved curriculum.

The professional mathematicians in SMSG believe that the mathematics presented in this text is valuable for all well-educated citizens in our society to know and that it is important for the precollege student to learn in preparation for advanced work in the field. At the same time, teachers in SMSG believe that it is presented in such a form that it can be readily grasped by students.

In most instances the material will have a familiar note, but the presentation and the point of view will be different. Some material will be entirely new to the traditional curriculum. This is as it should be, for mathematics is a living and an ever-growing subject, and not a dead and frozen product of antiquity. This healthy fusion of the old and the new should lead students to a better understanding of the basic concepts and structure of mathematics and provide a firmer foundation for understanding and use of mathematics in a scientific society.

It is not intended that this book be regarded as the only definitive way of presenting good mathematics to students at this level. Instead, it should be thought of as a sample of the kind of improved curriculum that we need and as a source of suggestions for the authors of commercial textbooks. It is sincerely hoped that these texts will lead the way toward a more meaningful teaching of Mathematics, the Queen and Servant of the Sciences.

CONTENTS

Chapter

1. INTRODUCTION TO FORMAL GEOMETRY 1
   - 1-1. Introduction 1
   - 1-2. Physical Space and Informal Geometry 1
   - 1-3. Geometrical Figures and a General Statement 2
   - 1-4. Inductive Reasoning and Informal Geometry 4
   - 1-5. Deductive Reasoning and Formal Geometry 10
   - 1-6. Definitions 15
   - 1-7. Special Words and Phrases 16
   - 1-8. Summary 18

2. SETS; POINTS, LINES, AND PLANES 19
   - 2-1. The Language of Sets 19
   - 2-2. Sets 22
   - 2-3. One-to-One Correspondence 30
   - 2-4. Points, Lines, and Planes 33
   - 2-5. Incidence Postulates--Points and Lines . . . 36
   - 2-6. Incidence Postulates--Points, Lines and Planes 42
   - 2-7. Three Theorems 47
   - 2-8. Summary 49
   - Review Problems 49

3. DISTANCE AND COORDINATE SYSTEMS 55
   - 3-1. Introduction 55
   - 3-2. The Set of Real Numbers 55
   - 3-3. The Ordering of the Real Numbers 61
   - 3-4. Distance 68
   - 3-5. Coordinate System on a Line 75
   - 3-6. Rays and Segments 82
   - 3-7. Interior Points 90
   - 3-8. Relationship Between Distances Relative to Diffenent Unit-Pairs 95
   - 3-9. Relationship Between Different Coordinate Systems on a Line 101
   - 3-10. Using a Given Coordinate System 107

Chapter

3-11. Length 114
3-12. Summary 121
Review Problems 123

4. ANGLES 133
4-1. Introduction 133
4-2. Separation 133
4-3. The Concept of an Angle 143
4-4. The Measurement of Angles 149
4-5. Ray Coordinate System in a Plane 158
4-6. Betweenness for Rays 164
4-7. Interior of an Angle 172
4-8. Right Angles and Perpendicularity 179
4-9. Supplements and Complements 189
4-10. Vertical Angles 194
4-11. Triangles and Quadrilaterals 201
4-12. Polygons 207
4-13. Dihedral Angles 215
4-14. Summary 217
Review Problems 219

5. CONGRUENCE 225
5-1. Introduction 225
5-2. Congruence of Triangles 226
5-3. Properties of Equality and Congruence . . . 233
5-4. Definitions in Proofs 241
5-5. Proofs in Two-Column Form 244
5-6. Some Experiments and Some Postulates 249
5-7. Writing Proofs Involving Triangle Congruences 260
5-8. Isoceles Triangle Theorem 275
5-9. Converses 280
5-10. Proving Non-Coplanar Triangles Congruent . . 284
5-11. Medians and Angle Bisectors 289
5-12. Using Congruences as a Mathematical Tool . . 291
5-13. Summary 299
Review Problems 300
Review Problems, Chapters 1 to 5 309

Chapter

6. PARALLELISM 315
  - 6-1. Introduction 315
  - 6-2. Definitions 316
  - 6-3. Indirect Method of Proof 325
  - 6-4. Parallel Line Theorems 330
  - 6-5. The Parallel Postulate 339
  - 6-6. Additional Parallel Line Theorems 341
  - 6-7. Parallelism for Segments and Rays 350
  - 6-8. Sum of the Measures of the Angles of a Triangle 358
  - 6-9. Right Triangles 366
  - 6-10. Inequalities in the Same Order 369
  - 6-11. Inequalities Involving Triangles 372
  - 6-12. Summary 381
  - Vocabulary List 383
  - Review Problems 384

7. SIMILARITY 391
  - 7-1. Introduction 391
  - 7-2. Proportionality 392
  - 7-3. Similarities Between Polygons 402
  - 7-4. An Experiment and a Postulate 411
  - 7-5. Triangle Similarity Theorems 417
  - 7-6. Similarities in Right Triangles 426
  - 7-7. The Pythagorean Theorem 433
  - 7-8. Special Right Triangles 438
  - 7-9. Summary 444

Appendix I. A Convenient Shorthand 451
Appendix II. Postulates for Addition and Multiplication 455
Appendix III. Theormes on Inequalities 461
Appendix IV. Rational and Irrational Numbers 465
Appendix V. How to Draw Pictures of Space Figures 471
Appendix VI. The A.S.A. and S.S.S. Congruence Postulates as Theorems 479

The Meaning and Use of Symbols 485
The Greek Alphabet 487
List of Chapters and Postulates 489
List of Theorems and Corollaries 493

Index

PREFACE

Formal geometry is usually introduced in a year-long course, often taught in the tenth grade. *Geometry with Coordinates* is designed for such an introductory course, and contains most of the material of geometry which has been found appropriate for a course at this level. The presentation here has one principal feature of novelty--it relies more heavily on algebra than has been common in a first course in formal geometry.

Such a fusing of geometry with algebra is characteristic of the development of mathematics, in which boundary lines among branches of the subject become blurred and eventually erased. Moreover, the relationship between geometry and algebra has recently been emphasized in the SMSG text *Geometry*, where Birkhoff's postulates, which make reference to the real numbers, are exploited. This book carries on in the same spirit, making explicit use of coordinate methods to obtain geometrical results.

It should be noted that the reader is *not* required to have much facility in algebra to understand this book. It is expected that classes using this text will have had one year of algebra, and that the geometry course will provide an opportunity for students to review, consolidate, and extend their knowledge of algebra.

The primary emphasis in this treatment remains on geometry as a deductive system. It is hoped that our presentation will help to make clear that analytic geometry fits into this deductive system--that it is not just a set of recipes for solving problems, but is also a portion of mathematics, having its place in the organic structure which mathematics is today.

Chapter 1

INTRODUCTION TO FORMAL GEOMETRY

1-1. Introduction.

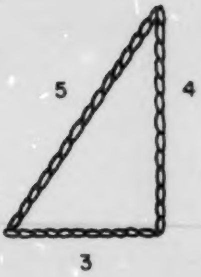
For several years now, in school and out, you have been acquiring knowledge about geometric objects like squares and circles. This knowledge was based on a study of things such as tiles and coins, or pictures of such things. These things and pictures are physical objects. This year we approach the subject of geometry from a mathematical rather than a physical point of view. A mathematician is concerned with ideas. For instance his idea of a square corresponds to, but is not the same as, a physical square. He uses pictures of squares to suggest not only what is true about them but also what may be true about geometric squares, that is, the squares which exist as ideas.

You may ask why a mathematician is not content to work only with physical objects. As you learn to work with geometric objects some of the reasons will become clear to you. A mathematician looks for relations among ideas. Knowing these relations gives him understanding and insight, and this kind of knowledge is satisfying. Moreover an understanding of relations helps him to discover additional properties which are less apparent in the physical world, and which are often found to be useful.

1-2. Physical Space and Informal Geometry.

Geometry began informally in the Babylonian and Egyptian civilizations as a physical science. Indeed, the very name of the subject refers to "earth-measurement"; and many results about areas of fields, volumes of buildings, and the like were obtained experimentally by observation of numerous examples.

For instance, the Egyptians learned by experience that if a figure like the adjacent one is made from sticks or stretched rope, with the lengths being 3 units, 4 units, and 5 units, then the figure has a "square corner." They used this fact to construct square-cornered buildings.



Other sets of numbers besides "3, 4, 5" will also give square corners. For example, the set

5, 12, 13

works just as well; so do

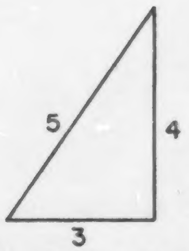
7, 24, 25 ;
8, 15, 17 ;
6.6, 8.8, 11 ;

and there are many more.

Starting with such experimentally verified facts, the Greeks brought order and organization to the subject by developing geometry formally as a logical system. In this chapter we compare the method of experimentation in informal or physical geometry with the method of logical deduction in formal geometry. The rest of the book is largely devoted to the formal development of geometry as a logical system.

1-3. Geometrical Figures and a General Statement.

First we make a "streamlined" version of the figure on the previous page by eliminating the fuzziness in the pictured rope. The features in which we are interested do not depend upon the sides being made of rope, or sticks, or anything of the kind. Ink marks on paper or chalk marks on a board will serve equally well. Moreover, we do not need to have "thick" marks (as in a picture of rope); "thin" marks will be even better. We are thus led to a picture like this one, in which



the sides appear thin and straight. But we realize that, however carefully a picture like this might be drawn, under a microscope it would look irregular and fuzzy, somewhat like the rope picture. To avoid these difficulties we form an idea of a triangle that exists only in our minds, not in physical space, and we refer to it as a geometric triangle. This geometric triangle has none of the irregularities of a physical triangle, but we find it convenient to use a diagram (that is, a physical triangle) as a reminder of what we wish to talk about.

It will be helpful to use the term right angle to correspond to "square corner." Now we search for some common feature of the various sets of numbers which, used as lengths of sides, lead to right-angled triangles. Do you know, or can you guess, what the common feature is?

It has to do with the squares of these numbers. Observe that

$$5^2 = 3^2 + 4^2 ;$$

likewise

$$13^2 = 5^2 + 12^2 .$$

You should convince yourself by checking that similar equalities hold for the other sets of numbers listed in Section 1-2.

From our experience with several special cases, we are thus led to a conjecture, which we term Statement A:

> Statement A: If the square of the length of one side of a triangle equals the sum of the squares of the lengths of the other two sides, then one of the angles of the triangle is a right angle.

Note carefully what this statement is about, and how far-reaching it is. Statement A does not refer to some rough physical triangles which we might make of sticks or stretched rope, nor to the somewhat finer (but still physical) triangles which we might sketch on a pad of paper or construct carefully with a draftsman's instruments. It refers to the ideal geometric triangles which exist in our minds. And it is not restricted to the handful of such triangles which are listed in Section 1-2. Statement A says something about every triangle which anyone might imagine, so long as the basic assumption is fulfilled (the square of the length of one side equals the sum of the squares of the lengths of the other two sides.) For every such triangle, whatever its size, one of the angles is a right angle, according to Statement A.

There are three major questions stemming from the foregoing discussion:

(1) What is the procedure by which we guess a general statement like Statement A?

(2) Can we be certain that it is true?

(3) Just what are the things being talked about in the statement? What are "sides," "triangles," "right angles," etc.?

We shall consider each of these questions separately in the following sections. But first an exploratory problem will help you understand our later analysis of questions (1) and (2).

Exploratory Problem: Let Q represent the expression $n^2 + n + 11$. Compute the values of Q for $n = 1, 2, 3, 4$. What features common to these values of Q do you notice? Do you think that these features of Q will still appear if larger values of n are used? Try $n = 5, 6, 7, 8, 9$. Do you think that these features will appear no matter how large n may be? Try $n = 10$.

1-4. Inductive Reasoning and Informal Geometry.

The exploratory problem and the discussion leading up to Statement A indicate an important method by which we guess statements that may have validity in general. We observe features common to a number of particular cases, and we formulate a statement that fits them all. This procedure is known as inductive reasoning, and is basic to work in the sciences. It is also widely used in mathematics and in everyday affairs. It is the method of informal geometry used in earlier grades.

Suppose, for example, that there are fifteen boys in your class, and that on one day each of the first five boys to arrive is wearing a tie. If you say to yourself, "I guess all the boys will be wearing ties today," you are generalizing from specific cases. Do you think that such a conclusion is entirely reliable? Would it be more reliable if each of the first ten boys to report is wearing a tie? Could you be certain of the result if you observed that the first fourteen boys to report are all wearing ties?

Clearly, inductive reasoning is never certain unless we can check every case. When we formulate a statement about all members of a set, we really mean all, not "almost all." Hence, one contradictory example proves the general statement false. Such a contradictory example is known as a counter-example. Thus the general statement,

> "All the boys in class will be
> wearing ties today,"

seems like a sensible inference, if the first five boys to report are wearing ties; it is even more likely, if the first ten, the first fourteen, boys are wearing ties; but it is proved false if the fifteenth boy reports without a tie.

Let us return to the exploratory problem. There are at least two fairly natural conjectures to make about the values of $Q = n^2 + n + 11$:

(X) If n is any positive integer (i.e., 1, 2, 3, etc.), Q is an odd number;

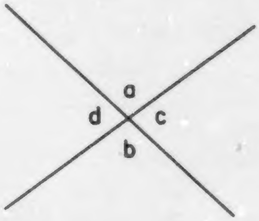
(Y) If n is any positive integer, Q is a prime number. (A prime number is an integer greater than 1 which has no positive factors except 1 and the number itself.)

You will find that both statements (X) and (Y) seem reasonable if you check $n = 1, 2, 3, \ldots, 9$. But for $n = 10$, $Q = 121$, and 121 is not a prime number. Hence, Statement (Y) is false, because we have discovered a counter-example.

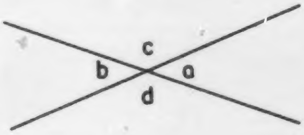
You can continue testing Statement (X) with larger and larger values of n without finding a counter-example. If Statement (X) still seems reasonable for all n up to 1,000,000 , does this mean that the statement is certainly true? No, for such a process does not tell us what might happen for $n = 1,000,001$. Some other procedure is necessary, and this is what we discuss in the next section.

Problem Set 1-4

1. (a) The diagram shows two intersecting lines. With your protractor, measure the angle labeled a , then measure the angle labeled b , and compare the sizes of the two angles. Now measure the two angles marked c and d , and compare their measurements.

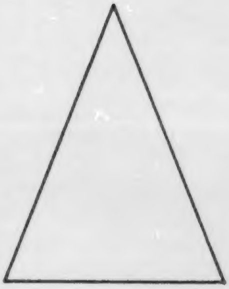


(b) This diagram shows another pair of intersecting lines. Repeat your experiment. Do you find a significant relationship between the number of degrees in angle a and in angle b ? How about the angles labeled c and d ?

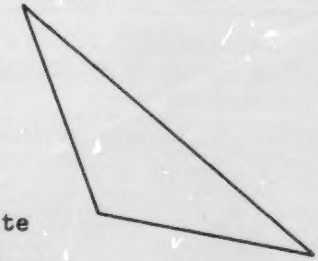


(c) Do you think that this relationship holds for every two lines which meet at a point? How would you express this fact in a short sentence but, as always, a complete sentence with good English structure?

2. (a) The triangle shown in the diagram has two sides of the same length. Use your protractor to measure each of the three angles of the triangle. Do two of the angles appear to have the same size? How are these two angles related to the two sides which have equal length?



(b) Repeat your experiment, using the triangle at the right. Which two sides of this triangle seem to have equal lengths? What can you say about the number of degrees in the two angles that are opposite the sides of equal length?



(c) Try to express the general idea which your experiment with the triangles has suggested to you. Be sure to use one or more complete sentences, giving all the necessary data first and then stating the conclusion.

3. In earlier grades you may have been asked to measure each angle in a triangle with your protractor. If so, you probably found that the total number of degrees in the three angles was approximately 180 . If you performed this experiment several times, what induction (that is, generalization) did you make?

4. Three quadrilaterals are shown. In each case, measure with your protractor each angle in the quadrilateral. By addition, find the total number of degrees in the four angles of each quadrilateral. Use induction to form a general statement. How can you achieve more reliability for the truth of your generalization?

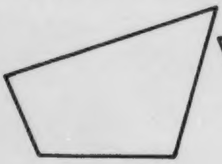


Figure a

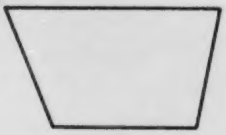


Figure b

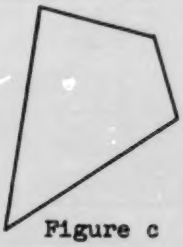
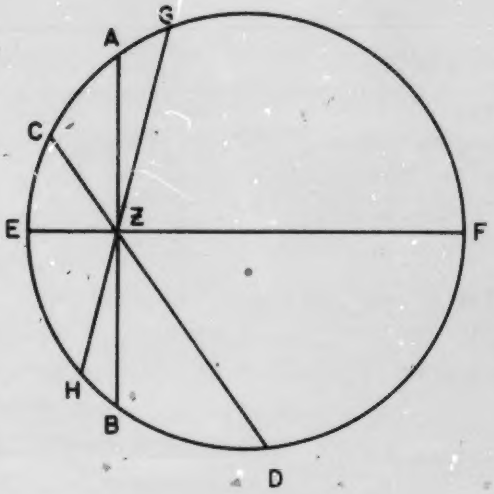


Figure c

5. In this circle there are four chords containing the point Z . Using a millimeter rule measure the



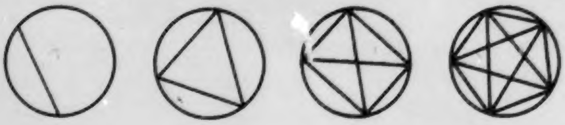
distance between A and Z ; between B and Z . Find the product of these measures. Now repeat this experiment for the distance between C and Z and between D and Z . Are the two products equal? Repeat for the distances between E and Z and between F and Z . Repeat also for the distances between H and Z and between G and Z . After making these pairs of measurements and finding the products what induction do you think is true?

6. Show that $n^2 - 2n + 2 = n$ if $n = 1$. Is the equation true when $n = 2$? Is it true for all values of n?

7. If two students carefully and independently measure the width of a classroom with rulers, one measuring from left to right and the other from right to left, they are likely to get different answers. You may check this with an experiment. Which of the following are plausible reasons for this?

(a) The rulers have different lengths.
(b) One person may have lost count of the number of feet in the width.
(c) Things are longer (or shorter) from left to right than from right to left.
(d) The errors made in changing the position of the ruler accumulate, and the sum of the small errors makes a discernible error.

8.



number of points connected	2	3	4	5	6
number of regions formed	2	4	8	16	?

Replace the question mark by the number you think belongs there. Verify your answer by making a drawing in which six points on a circle are connected in all possible ways.

9. Try to prove that the following statement is correct:

If n is any positive integer then
$Q = n^2 + n + 11$ is an odd number.

1-5. Deductive Reasoning and Formal Mathematics.

Suppose that, as a result of inductive reasoning, we have formulated a general statement about a set of objects. We think the statement is true, but how can we be sure? If the set is infinite, as is the case for most interesting and important generalizations in mathematics, we obviously can't check each particular instance. In such a situation we turn to deductive reasoning--we try by logical argument to deduce our conjecture from previously accepted statements. This is the characteristic method of formal geometry.

For example, from the two statements,

H_1 : All squares are rectangles,

and

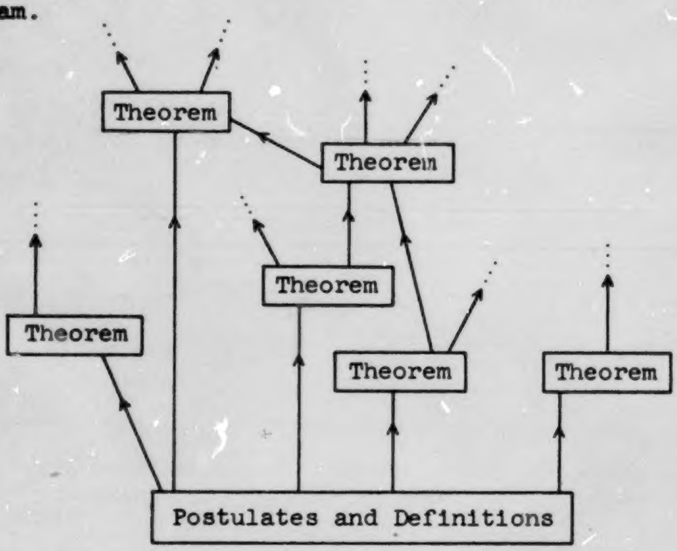
H_2 : No rectangle is a pentagon,

we can deduce the statement,

C : No square is a pentagon.

The combination of the two statements labeled H_1 and H_2 is called the hypothesis; the statement labeled C is called the conclusion. Thus, by deductive reasoning we obtain a conclusion from some hypotheses. But where do the hypotheses come from? We may be able to deduce them as conclusions from some other hypotheses, possibly simpler ones. And where would these other hypotheses come from? Can you see that this process of pushing our argument ever farther back must stop somewhere? We decide to make a start somewhere by frankly assuming some hypotheses.

These basic initial assumptions are usually called postulates or axioms. We choose as postulates some relatively simple statements which seem reasonable on the basis of our experience. In this book we shall assume over 20 postulates as the foundation for building the logical system of geometry. Other statements in the geometry, called theorems, will be proved (i.e., logically deduced) from these postulates. Once we have proved a theorem, we can use it, as well as the postulates, in the logical arguments by which we deduce more theorems. In this fashion we obtain a satisfying organization of the subject of geometry--we no longer deal with a jumble of "facts," but we can see how the "facts" fit together, like the bricks in a building. In proving a theorem, thus adding to our organization, we shall use only postulates we have assumed, definitions we have formulated, and theorems we have proved. The growth of a logical system is suggested by the following diagram.



Let us return to a consideration of our postulates, those assumptions on which the rest of our geometry is based. Are we certain that they are true? The answer is "no"--we just decide to assume them. Then what about the theorems, our conclusions by logical deduction from the postulates? We can't say that they are "true," either--the best we can do is to say that they are valid deductions from the postulates. If the postulates are true, then valid deductions from them are also true.

As an example of deductive reasoning, we shall prove Statement (X) in Section 1-4, taking as hypotheses the familiar facts of arithmetic.

We are, then, to prove the following statement:

(X) If n is any positive integer, and Q is given by $n^2 + n + 11$, then Q is an odd number.

Proof: We can write Q as follows:

$$Q = n(n + 1) + 11 .$$

Since n and $(n + 1)$ are consecutive integers, one of them is an even number. Hence, $n(n + 1)$ is even, for the product of two integers, one of which is even, is an even number. But the sum of an even number and an odd number (in this case, the number 11) is odd. Thus we have deduced that Q is odd, no matter what integer n is.

We work out another example of deductive reasoning in order to illustrate the important concept of indirect argument:

In a certain small community consisting exclusively of young married couples and their small children, the following facts are known to be true.

(a) Every boy has a sister.
(b) There are more boys than girls.
(c) There are more adults than children.

Prove that there must be at least one childless couple.

Here our initial assumptions are simply the facts stated in the problem. This time, the logical process which will lead us from the given information to the desired conclusion is what is known as indirect reasoning. In this, we assume the opposite of what we really want to prove, and show that this is impossible. This leaves the desired conclusion as the only remaining possibility.

We begin, then, by assuming that there is no childless couple. From this, we conclude that every family must have at least one girl, since by the first fact there can be no family having only boys. Thus, there are at least as many girls as there are families. Moreover, by the second fact, there are actually more boys than there are families. Hence the number of boys and girls together is more than twice the number of families. But this means that there are more children than adults, which contradicts the third fact. Hence it is false that every family has a child. In other words, at least one family must be childless, which is what we were asked to prove.

Problem Set 1-5

1. Consider each pair of sentences below as a pair of hypotheses, the first one general, the second specific. If the specific hypothesis is related to the general one so that a logical deduction follows, state that deduction. If one does not follow, explain why.

   (a) If a student is in Miss Smith's class, Miss Smith is his teacher.

   John is in Miss Smith's fifth period English class.

   (b) Every member of the High Peak Club must have climbed a mountain.

   My father is a member of the High Peak Club.

   (c) To be a policeman in Elk City one must be at least 6 ft. tall.

   Jim's uncle is a policeman in Elk City.

   (d) All Eagle Scouts must have passed Test R.

   Harry is an Eagle Scout.

   (e) All seniors in the school left for the beach Saturday morning.

   Alice went to the beach Saturday morning.

   (f) Children under 12 years of age ride on buses for half fare.

   Jack rides for half fare.

   (g) All insects have six legs.

   A fly has six legs.

(h) Rainy days are disagreeable.

Friday was a rainy day.

(i) All apples are red when ripe.

The Early Transparent is a variety of apple.

(j) All trees have needles.

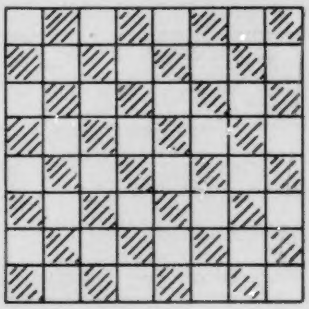
The fir is a kind of tree.

2. Prove each of the following statements.

(a) In a classroom of 25 students, there are at least two whose birthdays fall in the same month.

(b) There are at least two trees on this earth that have the same number of leaves. (Assume that the number of trees on earth is larger than the number of leaves on any one tree and that every tree has at least one leaf.)

3. At the right is a checkerboard. As you can see, it has 64 squares; it is to be covered by dominoes each the size of two of these squares. Of course, it takes 32 such dominoes to cover the entire board without any overlapping. Now suppose that the two black squares, one at each end of a diagonal are removed from the board. Can what remains of the board be covered by 31 dominoes? Explain.



4. Give a conclusion which might have been drawn from each of the following situations or statements. Indicate whether the conclusion is a deduction or an induction.

(a) In the ten years in which I have lived in this part of the country we have never had a snowstorm.

(b) The price range for a suit in that store is $35 to $150 .

(c) In the thousands of specimens of that type of cell which I have examined all have had a thick cell wall.

(d) All the buildings designed by Mr. Brown are the new steel and glass type of structure.

Mr. Brown was the architect for the medical center in Springdale.

(e) All officers of the High School Student Body must have passing grades.

Jerry is the secretary of the Student Body.

1-6. Definitions.

The triangles with which we dealt in Section 1-2 are known as right triangles. We give a formal definition as follows:

DEFINITION: A triangle that has a right angle is a right triangle.

In this definition we underline "right triangle" to show what is being defined. The purpose of the definition is to enable us to replace the phrase, "a triangle that has a right angle," by a shorter phrase, "a right triangle," and thus avoid the inconvenience of having to repeat the long phrase in a discussion about right triangles. Since the two phrases are names for the same object the definition may be reversed as follows:

DEFINITION: A right triangle is a triangle one of whose angles is a right angle.

The first form of the definition might also have been written:

If a triangle has a right angle,
then it is a right triangle.

The second form of the definition might also have been written:

If a triangle is a right triangle,
then it has a right angle.

We can write both of these together as follows:

DEFINITION: A triangle is a right triangle if and only if one of its angles is a right angle.

The definition of right triangle given above might be considered to be useless, for it appears to raise more questions than it answers. What is a triangle? What is an angle? What is a right angle? We will find in Chapter 4 that angle is defined in terms of ray and some other concepts. What, then, is a ray? We will find in Chapter 3 that ray is defined in terms of point, line, and some other concepts. What, then, is a point?

Do you see that we have a situation similar to what we encountered in Section 1-5, where we discussed the necessity of assuming some postulates in order to make a start with our deductive system? Just as we can't prove all statements and hence must assume some, so likewise we can't define all terms and hence must accept some as undefined.

This may seem like an unsatisfactory procedure, but it really works very well. In this book we shall accept "point," "line," and "plane" as undefined terms, and will discuss the process of definition more extensively in Chapter 2 where we encounter these words formally, and again in Chapter 5.

1-7. Special Words and Phrases.

In mathematics we sometimes use ordinary words in special ways. We try to be careful and precise in our language, which is not always the case in everyday discourse. We have already noted that the word "all" really means "all," and not merely "a great many" or "most." When we state that there is a thing of a certain kind, we mean that there is at least one such, but we do not mean to exclude the possibility of there being more than one. For instance, we might say, "There is a line which contains point A ." If we want to call attention to the fact that there is exactly one thing of a certain kind, we use the phrase one and only one and sometimes say that the object in question is unique. For instance, we might say, "There is one and only one line which contains two given points," or we might say, "There is a unique line which contains two given points." Still another way to put this is to state: "Two given points determine a line."

Many of our statements have the form, "If P , then Q ," where P and Q are statements. For example, an important statement, known as the Pythagorean Theorem, can be worded as follows:

> "If one angle of a triangle is a right angle, then the square of the length of one side of the triangle equals the sum of the squares of the lengths of the other two sides."

You will recognize that our Statement A of Section 1-3 also has the if-then form and that it can be obtained from the statement of the Pythagorean Theorem by interchanging the statement of the if-clause with the statement of the then-clause. It happens that both Statement A and the Pythagorean Theorem can be deduced from our postulates of geometry. Each is therefore an example of a theorem. You will find that many theorems in geometry are stated, or can be stated, in the if-then form. This will be useful to us because this form makes it easy to identify the hypothesis and the conclusion. The hypothesis is its if-clause and the conclusion is its then-clause. Reasoning by deduction, the method of formal geometry starts with the hypothesis and proceeds logically through a sequence of statements until the conclusion is obtained. Such a sequence of statements is a mathematical proof. In our next chapter we shall see some examples of such proofs.

1-8. Summary.

We have stressed the contrasts between informal geometry, which deals with certain properties of objects we can see and touch, and formal geometry, which involves similar properties of objects which we can only imagine.

Within formal geometry (the subject matter of this book), there are also several interesting contrasts:

(1) between inductive reasoning, which we use to help us discover geometrical relationships, and deductive reasoning, which we use in proving these relationships;

(2) between postulates (assumed statements) and theorems (deduced statements);

(3) between undefined terms and defined terms.

It will be helpful if you keep these contrasts in mind as we begin our study of formal geometry.

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If you would like to learn more about the ideas discussed in this chapter, it is suggested that you read some of the following books, or chapters selected from them.

Bell, E. T., *Men of Mathematics*. New York: Simon and Schuster, 1937.

Eves, H., *An Introduction to the History of Mathematics*. New York: Rinehart and Company, 1953.

Kline, M., *Mathematics in Western Culture*. New York: Oxford University Press, 1953.

Kramer, E. E., *The Main Stream of Mathematics*. New York: Oxford University Press, 1951.

Sanford, Vera A., *A Short History of Mathematics*. Boston: Houghton, Mifflin and Company, 1930.

Struik, D. J., *A Concise History of Mathematics*. 2 Volumes. New York: Dover Publications, 1948.

Turnbull, A. W., *The Great Mathematicians*. New York: New York University Press, 1961.

Chapter 2

SETS; POINTS, LINES, AND PLANES

2-1. *The Language of Sets.*

In our study of geometry we shall have to make extensive use of a number of technical terms whose precise meanings we must understand clearly. It would of course be a serious mistake to burden ourselves with all of these at the beginning of our work and we shall introduce them only as we have need of them. However, there are a few which we need to start with, and the most fundamental of these is, perhaps surprisingly, the simple word *set*.

The idea of a set is a very familiar one. A basketball team on the court is a set of five players, a baseball team in the field is also a set of players. In each case the players are the *members* of the set. However, the members of a set need not be people. Thus the United States is a set whose members are the fifty states, and the English alphabet is a set whose members are the twenty-six letters. We even have such explicit everyday uses of the word as "a set of china," "a set of silverware," or a "set of golf clubs." Still another example is the set whose five members are the Empire State Building, Mount Fujiyama, Abraham Lincoln, the Declaration of Independence and the North Star.

The members of a set are often called the *elements* of the set. The members, or elements, of a set are said to *belong to* the set, and the set is said to *contain* its elements and *consists* of all its elements. The letter f belongs to the English alphabet, a certain set of china contains a meat platter. The set composed of the four numbers 3,5,7,11 is often denoted by the symbol {3,5,7,11} . The notation {Alaska, California, Hawaii, Oregon, Washington} names the set of states which border on the Pacific Ocean.

Do the sets {3,5,7,11} and {5,11,7,3} have the same elements? Do you think that the set {3,5,11} is the same as the set {2,5,11} ? Why, or why not? Would you say that the set {5,4,3,2} has the same members as the set {5,3,2} ? Do you believe that a set with fifty members can be the same as a set with sixty members?

Let us agree that two sets are the same provided each of them has exactly the same members as the other. If the two sets are named A and B , instead of saying that the sets A and B are the same, we often say they are equal, and we write $A = B$. For example, the sets {3,5,7,11} and {5,11,7,3} are equal, and we express this sameness by writing {3,5,7,11} = {5,11,7,3} . Explain why the statement {2,4} = {5,2} is false. Is the statement {-1,0,1} = {1,-1} true or false? Why?

Often a certain set may be described in different ways. If S is the set whose elements are the integers between 1.5 and 6.2 , and if M is a short name for the set {2,3,4,5,6} , then $S = M$.

When you were asked in algebra to solve the equation $x^2 - 5x + 6 = 0$, your problem was essentially to find the members of the set of solutions of the equation. Since $x^2 - 5x + 6 = (x - 2)(x - 3)$, you concluded that the solution set is {2,3} . This set is the same as the set whose elements are the smallest two primes.

Problem Set 2-1

1. Fill in each blank with an appropriate word:
Canada ______ to the set of nations called the British Commonwealth. The senior United States Senator from your state is a _____ of the set of public officials. The set {1,2,3,6,7,8} ______ the element 6 .

2. Name each of the following sets by making a list of the members and enclosing the list in a pair of braces: $\{\ \}$.

   (a) The set of members in your family.
   (b) The set of courses in which you are presently enrolled.
   (c) The set of positive even integers from 2 to 14 , inclusive.
   (d) The set of all those numbers from 1 to 50 , inclusive, which are squares of integers.

3. (a) List the set of integers from 0 to 10 which are perfect squares.
   (b) Is the set $\{4,9\}$ equal to your answer for (a) ? Why?

4. How might the set $\{3,5,7,9\}$ be described in words? Is it possible to give more than one description?

5. Which of the sets listed below are the same:

   (a) the set of integers from 1 to 5 , inclusive;
   (b) the set of positive integers between -6 and 6 ;
   (c) the set $A = \{0,1,2,3,4,5\}$?

6. Are the sets {John, *,$,5,t} and {$,t,John,*,5} equal or not? Why?

7. (a) Given that x stands for such a number that $x^2 = 81$, find one possible value for x . Is there any other number which x might stand for? What is the solution set of the equation $x^2 = 81$?
   (b) Write the solution set of the equation $x^2 + 5 = 21$.

8. Write the solution set of each of the following equations:

   (a) $5x - 3 = 12$
   (b) $4(2x - 3) = 8$
   (c) $2(3x - 4) = x + 10$
   (d) $x^2 - 7 = 18$
   (e) $x^2 - 3x - 10 = 0$

9. List at least three members of each set described below. In each case state whether or not it would be possible to make a complete list of all the members.

   (a) the even integers from 10 to 200
   (b) the positive integers between -10 and 200
   (c) the rational numbers between 1 and 2
   (d) the integers
   (e) the integers divisible by 7
   (f) the positive integral factors of 30
   (g) the prime factors of 30

*10. Consider the two sets $X = \{6, \sqrt{2}, -\frac{1}{3}, 108.29\}$ and $Y = \{\sqrt{2}, -81, 0, -\frac{1}{3}, 108.28\}$.

   (a) What elements belong to both X and Y ?
   (b) What elements belong to one of the sets X and Y , but not to both?
   (c) What elements belong to either X or Y or both?

2-2. Sets.

We observe that the set $\{3,6,7,9\}$ is not the same as the set $\{3,7,9\}$. The two sets are not equal because the number 6 appears in one membership list, but not in the other. However, each element of $\{3,7,9\}$ is also a member of the other. One of the sets seems to be a part of the other. We are led in this manner to the notion of a subset.

If each element of a set A is also a member of a set B , we may say that A is a subset of B . Other ways of expressing the same idea are: the set B contains the set A , or A is contained in B .

For instance, {Paris, Rome} is a subset of {London, Paris, Rome, Tokyo} , and $\{-1,0,1\}$ contains the set $\{-1,1\}$. Notice that the word "contains" has two usages: a set may contain a member, and a set may contain a subset.

Although the notions of a part of a set led us to introduce "subset" and although the prefix "sub-" may emphasize the "part of" idea, you should observe carefully that the definition of "subset" does not require that A and B be different. Since every element of any set is also a member of the same set, every set is a subset of itself.

It is not always possible to list each and every member of a set. The even numbers, 2, 4, 6, ... (the three dots are read "and so on indefinitely") form a subset of the set of all positive integers 1, 2, 3, 4, ... The latter, {1,2,3,4,...}, consisting of all positive integers, is, in turn, a subset of {0,1,-1,2,-2,3,-3, ...} consisting of all integers. The members of Congress form a subset of the set of all United States citizens. The set consisting of just this one book is a subset of the set of all books.

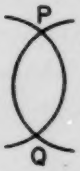
The last illustration suggests the need for distinguishing carefully between a set and the elements which it contains. When a set contains more than one element, this distinction is clear. Our notation with braces helps to emphasize the situation: {3,5,7,11} represents a set; its members are 3, 5, 7, 11 . What notation would you use to name the set whose only element is 8 ? Do you see any distinction between the number 8 and the set {8} ?

When a set contains a single element, it may seem trivial or even unreasonable, to think of the element and the set consisting of just one element as two quite different things. But there really is an important distinction. Perhaps the best way to explain the difference is by an example. In a certain high school, Mary Brown was the only student who enrolled in the course in advanced Latin. In other words, at the beginning of the term the advanced Latin class was a set consisting of the single element, Mary Brown. Because teachers were scarce that year, the principal decided that he couldn't continue a class with just one student, so the class was canceled. Thus the set consisting of Mary Brown was eliminated, its existence ended by order of the principal. On the other hand, Mary Brown was certainly not eliminated. To have eliminated the set consisting of Mary Brown was a necessary and desirable economy. To have eliminated Mary Brown would have been murder!

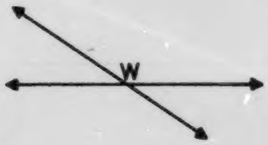
Consider the two sets {1,3,4,5,9} and {3,4,6,8} . Is there a number which belongs to one of these sets but not to the other? Are there several such elements? Is there any number which these two sets have in common, that is, any number which is an element of each set? Each of the numbers 3 and 4 belongs to both sets. We say that the set {3,4} , which is composed of the numbers common to the given sets, is the intersection of the given sets. In general, if we have any two sets, their intersection is the set consisting of all common members. In other words, the intersection is composed of all elements which belong to each of the given sets.

One set consists of the New England states; another set consists of the states whose names begin with the letter M ; the intersection of these two sets is {Maine, Massachusetts} . What is the intersection of the set of all blue-eyed people and the set of all girls in your geometry class? What is the intersection of the set of prime numbers and the set of natural numbers between 4 and 12 ?

The word "intersection" has been chosen in the language of sets, because of its usage in geometrical situations. If we imagine that the two circular arcs in the diagram are sets of points, then their intersection is the set composed of the two points labeled P and Q . As an exercise in the proper use of set notation, write a symbol for this intersection.



Likewise, denote the intersection of the two lines suggested in the picture.



Given two sets, we sometimes wish to put them together, so to speak, by forming a set which contains all the elements of the given sets. For instance, if the sets {1,2,3} and {2,3,5,7} are put together in this manner, we obtain the set {1,2,3,5,7} . Notice that the element 2 , which is common to the given sets, appears as an element of the new set only once. A similar remark applies to the common member 3 . The set we have built in the example is called the union of the two given sets. In general, if we have any two sets, their union is the set consisting of all elements which belong to one or the other, or both, of the given sets.

The union of the set of all fathers and the set of all mothers is the set of all parents. The union of the set of odd integers and the set of even integers is the set of integers.

Problem Set 2-2a

1. Fill in each blank:

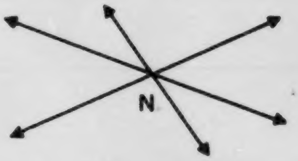
   (a) The _____ of {a,d,j} and {j,e,m} is {a,m,e,j,d} .

   (b) The _____ of {-2,-1,0,1,2} and {0,2,4,6,8} is {0,2} .

   (c) The set {0,2} is _____ in {-2,-1,0,1,2} .

2. Consider more than two sets, say the sets {1,2,4} and {2,4,7} , and {1,7,8} . If you used the members of these sets to form the set {1,2,4,7,8} , what name might you give to the set you formed? You might call it the _____ of the three given sets. (Fill in the blank.)

3. Imagine that each of the three intersecting lines represented in the diagram is a set of points. What set is the intersection of all three lines?

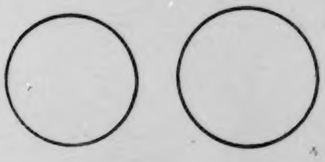


4. Given A = {2,4,6} , write (in set notation) seven subsets of A . A proper subset of a set is contained in but does not equal the set. List at least five proper subsets of A .

5. Make an example in which you name four sets (each containing at least 3 elements) such that the intersection of any two of these sets contains at least one member. Then write the set which is the union of the four sets.

In general, we often speak of the union, or the intersection, of more than two sets. The intersection consists of the elements common to all of the sets. The union consists of all elements which are members of any one or more of the given sets.

Sometimes the notion of intersection of sets leads us to a strange situation. Suppose we are asked to tell the intersection of the set {2,5,8} and the set {3,7} . Or suppose we are asked about the intersection of the two circles shown in the diagram.



In each of these illustrations, the two given sets have no common elements. Can we nevertheless consider that two such sets have an intersection and that the intersection is a set? That is, can we speak about a set containing no elements at all? This may seem like a foolish question but it really is not. In many mathematical situations it is a great convenience to be able to answer it affirmatively. If you recall the example about Mary Brown and the advanced Latin class, you should realize that a set is quite a different thing from the elements which it contains (or doesn't contain!). With this in mind, we recognize as a meaningful idea the notion of the empty set, a set consisting of no elements at all. Now we can say that the intersection of {2,5,8} and {3,7} is indeed a set; in fact, it is the empty set. Likewise the set consisting of the numbers which satisfy the equation $x + 1 = x - 1$ is the empty set, for no number is a solution of the equation.

The use of the empty set permits us to speak of the intersection of two sets whether the sets have any elements in common or not. However, it will often be convenient to have a way of indicating that two sets actually have at least one element in common, that is, that their intersection is not the empty set. This we shall do by saying that the two sets intersect. Briefly, the intersection of two sets may be the empty set, but if two sets intersect there is at least one element which belongs to both of them.

Problem Set 2-2b

1. Let A be the set $\{3,5,6,9,11,12\}$ and B be the set $\{4,5,7,9,10,11\}$. Give the answer to each of the following questions, by using the brace notation and listing the numbers. What is the intersection of the sets A and B ? What is the union of A and B ?

2. Consider the following sets:

   S_1 is the set of all students in your school.

   S_2 is the set of all boys in your student body.

   S_3 is the set of all girls in your student body.

   S_4 is the set of all members of the faculty of your school.

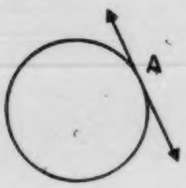
   S_5 is the set whose only member is yourself, a student in your school.

   (a) Which pairs of the above sets intersect?
   (b) Which set is the union of S_2 and S_3 ?
   (c) Which set is the union of S_1 and S_5 ?
   (d) Describe the union of S_1 and S_4 .
   (e) Which of the sets are subsets of S_1 ?

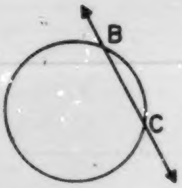
3. Consider the following three sets: $L = \{r,s,t\}$, $M = \{t,u,v\}$, $N = \{v,w,x\}$.

   (a) Which pairs of these sets intersect?
   (b) Describe the intersection of sets L and M ; of sets L and N .

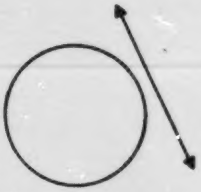
4. Consider the following three diagrams.



CASE I



CASE II



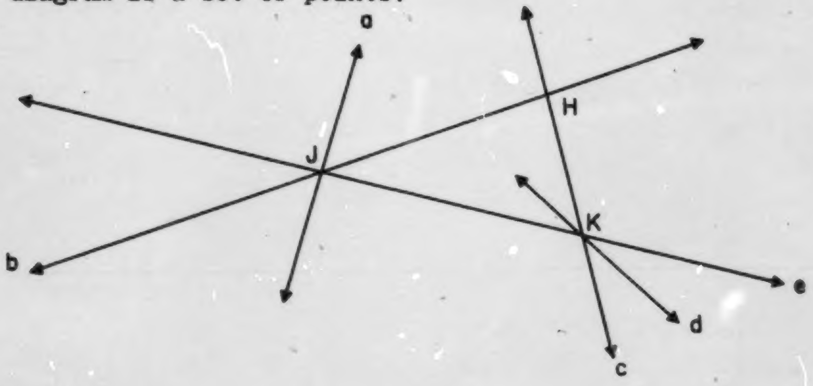
CASE III

(a) In each case if you consider the points on the line and the points on the circle as forming two sets of points, what is their intersection?

(b) In Case II, how would your answer be changed (if at all), if you considered one set to consist of all the points either on or inside the circle?

(c) In Cases I and III, how would your answer be changed (if at all), if you made the modification mentioned in Part (b)?

5. Imagine that each of the five lines suggested in the diagram is a set of points.



(a) Fill the two blanks: The set {H} is the intersection of line _____ and line _____ .

(b) What is the intersection of a and e ?

(c) What is the intersection of the three sets c, d, e ?

(d) Fill the blank with a word: The intersection of the sets a and b and the intersection of b and e are _____ .

(e) What is the intersection of the four sets a, b, c, d ?

6. Consider the set of all positive integers divisible by 2 and the set of all positive integers divisible by 3 .

   (a) Describe the intersection of these two sets. List its four smallest members.
   (b) Describe the union of the two sets. List its eight smallest members.

7. Consider the set of integers between 10 and 100 , written as usual, in decimal notation. What is the intersection of the following four subsets?

   (a) The subset consisting of prime numbers.
   (b) The subset consisting of numbers whose representation includes the digit 7 .
   (c) The subset consisting of numbers for which the sum of the digits is an even number.
   (d) The subset consisting of numbers in which the units digit is greater than the tens digit.

8. For convenience mathematicians often use the symbol $\cap$ to mean "intersection" and the symbol $\cup$ to mean "union." Given $M = \{a,b,c,d,e\}$ and $N = \{e,f,a,g\}$.

   (a) What is meant by the symbolic statement $M \cap N$; $N \cup N$?
   (b) List in set notation the members of the set $M \cap N$; $M \cup N$.

9. The empty set has been defined as the set consisting of no elements. This is sometimes called the null set. The symbol $\emptyset$ is often used to denote the empty set. For example, if $A = \{p,q,r\}$ and $B = \{m,n,o\}$, then $A \cap B = \emptyset$. (Note that we do not use braces in writing the symbol for the empty set.)

   (a) If D is the set of males who are older than 18 years and E is the set of males who are 18 years old or younger, what is $D \cap E$? What is $D \cup E$?

(b) If $A = \{1,3,5,7,9,11, \ldots\}$, that is, if A is the set of all odd natural numbers, and if $B = \{2,4,6,8,10, \ldots\}$, express $A \cup B$ in set notation. Express $A \cap B$ in set notation.

10. Let $f = \{3,6,9,12,15,18\}$, $g = \{7,14,21,28\}$, $h = \{6,12,14,28\}$. Using the brace notation, write the sets represented by:

(a)	$f \cup g$	(c)	$f \cap (g \cup h)$
(b)	$f \cap g$	(d)	$(f \cap g) \cup (f \cap h)$

2-3. One-to-one Correspondence.

In a classroom of 20 seats there are exactly 20 students. For each student there is a seat and for each seat there is a student. When each student is assigned to a particular seat we have an example of a *one-to-one correspondence* between the set of students and the set of seats. Of course, if another student were admitted to the class then we could no longer have a one-to-one correspondence between the set of seats and the new set of students. If a new seating arrangement were made for the set of 20 students then we would have a different one-to-one correspondence. We note that in order to have a clear notion of a one-to-one correspondence between two sets, we must know how the elements of the two sets are matched. We call a pair of elements that are matched a pair of *corresponding* elements. We record a corresponding pair with the symbol $\longleftrightarrow$.

Example 1. Below are three possible one-to-one correspondences between {Jim, John, Jane} and {Buffalo, Boston, Baltimore} .

$$C_1\begin{cases}\text{Jim} \longleftrightarrow \text{Buffalo}\\ \text{John} \longleftrightarrow \text{Boston}\\ \text{Jane} \longleftrightarrow \text{Baltimore}\end{cases}\quad C_2\begin{cases}\text{Jim} \longleftrightarrow \text{Boston}\\ \text{John} \longleftrightarrow \text{Baltimore}\\ \text{Jane} \longleftrightarrow \text{Buffalo}\end{cases}\quad C_3\begin{cases}\text{Jim} \longleftrightarrow \text{Baltimore}\\ \text{John} \longleftrightarrow \text{Boston}\\ \text{Jane} \longleftrightarrow \text{Buffalo}\end{cases}$$

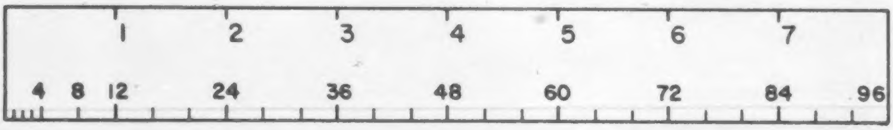
Find another one-to-one correspondence between the two sets.

Example 2. Consider

$$\begin{cases} \text{Jim} \longleftrightarrow \text{Buffalo} \\ \text{Jane} \longleftrightarrow \text{Baltimore} \\ \text{John} \longleftrightarrow \text{Boston} . \end{cases}$$

Is this one-to-one correspondence the same as C_1 ? Write C_1 in still another way.

Example 3. Below is a picture of an "eight-foot rule" marked in feet on the top edge and in inches on the bottom edge.



You can see that this rule matches 3 on the top scale with 36 on the bottom scale, and 5 on the top scale with 60 on the bottom scale. With what number of the bottom scale does 6 on the top scale match? What number on the top scale is matched with 6 on the bottom scale? Do you see that this ruler establishes a one-to-one correspondence between the set of all real numbers between 0 and 8 and the set of all real numbers between 0 and 96 ?

Problem Set 2-3

1. Let $\{A,B,D\}$ and $\{X,Y,Z\}$ be two sets. A one-to-one correspondence between the set $\{A,B,D\}$ and the set $\{X,Y,Z\}$ is

$$C_1 \begin{cases} A \longleftrightarrow X \\ B \longleftrightarrow Z \\ D \longleftrightarrow Y . \end{cases}$$

(a) Give a different one-to-one correspondence between $\{A,B,D\}$ and $\{X,Y,Z\}$.

(b) Give still another one-to-one correspondence between the two sets.

2. Give three different one-to-one correspondences between the set $\{A,B,D,E,G\}$ and the set $\{U,V,X,Y,W\}$.

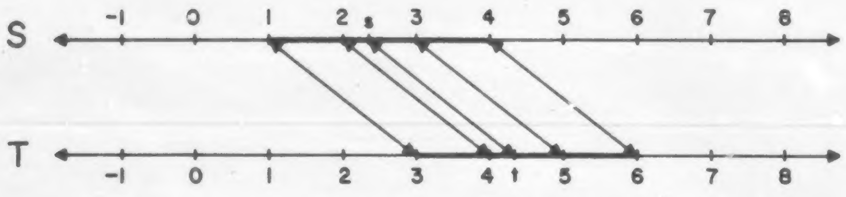
3. Can there be a one-to-one correspondence between the set $\{-1,0,1\}$ and the set $\{4,7\}$? Explain your answer.

4. Let N be the set of all negative numbers and let P be the set of all positive numbers. With each negative number n , match a positive number p , such that $n + p = 0$.

   (a) What number matches with -7 ?
   (b) With what negative number is $\sqrt{2}$ matched?
   (c) With what number is $\sqrt{5} - 2$ matched?
   (d) Does every member of P match with some negative number?
   (e) What negative number corresponds to the positive number p ?
   (f) Does any member of N correspond to more than one element of P ? Why?
   (g) Does any element of P correspond to more than one member of N ? Why?
   (h) Is the matching described above a one-to-one correspondence between the set N of all negative numbers and the set P of all positive numbers?

5. Let S be the set of all real numbers between 1 and 4 , inclusive. Let T be the set of all real numbers between 3 and 6 , inclusive. Consider the correspondence between S and T , given by

$$s \longleftrightarrow t \text{ , if and only if } t = s + 2$$



(a) What element of T corresponds to the number 2.3 in S ?

(b) What element of S corresponds to the number 5.3 in T ?

(c) What element of S corresponds to the number t in T ? Give your answer in terms of t .

(d) Does the smallest number in S correspond to the smallest number in T ?

(e) Does each member of S correspond to exactly one number in T ?

(f) Does each member of T correspond to exactly one number in S ?

(g) Is the correspondence a one-to-one correspondence between the set S and the set T ?

6. Let S be any nonempty set. Consider the correspondence between the set S and itself which assigns to each member of S the same member:

$$s \longleftrightarrow s \ .$$

Explain why this is a one-to-one correspondence. (We call this correspondence the "identity correspondence on the set S" .)

2-4. Points, Lines, and Planes.

We are now in a position to begin our development of geometry. As we saw in our brief discussion of the deductive method of reasoning in Chapter 1, this requires that we agree on certain undefined terms and also on certain postulates, or unproved statements, about the undefined terms. We can then, by deduction, derive more and more properties of the objects we are studying.

The fundamental undefined terms in our system are point, line and plane. We all have some understanding of what these words mean in the physical world. We have often talked about them in the past, and have drawn pictures (as we did in Chapter 1, and will do again in later chapters) to suggest our ideas about them. However, attempted definitions of these terms would turn out ultimately to depend upon the terms themselves (i.e., the definitions would be "circular"), or else would depend upon other words which would have to be taken as undefined. You can find many examples of "circular" definitions. One student, wanting to know the definition of the word "dimension," consulted the dictionary. To understand the dictionary definition he had to know the meaning of "size" or "measure." He therefore looked up "size" and "measure" and found that their definitions used the word "dimension" again! To get meaning from a sequence of definitions, somewhere in the cycle you must cut in by knowing what a word means not because of a definition but because your experience provides you with a meaning for it.

Rather than giving inadequate, or even meaningless, definitions of point, line and plane, we take the bold step of attempting no definitions of these terms, letting our postulates give them the meaning we wish them to have.

This is not as strange or unusual a procedure as it may seem at first. Consider, for instance, the game of chess. It is played with various pieces, which together form a set of chessmen. In some sets the men are black and white, in some they are red and white; in still others they are red and black. In some expensive sets they are made of ivory. In ordinary sets they are made of wood or plastic. They come in a great variety of sizes and designs. How then shall we define one of the pieces in a chess set? Surely not by its physical characteristics. The only features which really matter are its properties, that is, the things it can do, the way it moves, the way it can capture other pieces. And these are prescribed completely by the rules of the game, which are accepted without argument, or "proof," by those who play the game. Indeed some experts are able to play chess without any chessmen at all! The only thing of significance to them is the idea of the pieces and their properties. Most of us, however, are unable to keep in mind the intricate set of relations that exist among the various pieces throughout a game of chess. We need physical representations of the chessmen to remind us of their abstract properties and to suggest further relations to be explored. Points, lines and planes are somewhat like the pieces in a game of chess. Their real significance comes from the "rules of the game," that is, the postulates we agree to accept when we study geometry.

In this study, most of us will need the help of concrete objects to remind us of the ideas we have in our minds. For this reason we will make little dots and call them "points" and draw long marks and call them "lines." However, these physical marks are not the same as our mathematical concepts. We must remember that while such pictures can suggest relationships to us, our conclusions must be obtained by deduction from our postulates in order to be accepted.

Altogether we shall need about 30 postulates as a basis for the geometry we intend to develop. Some of them are more complex than others and can appropriately be introduced only after the simpler ones and their consequences have been explored. In the next section we shall begin our study by examining those postulates which describe the most elementary properties of points, lines, and planes, namely the ways in which they are determined and what their intersection may be. We call these the incidence relations.

Problem Set 2-4

Listed below is a set of statements about imaginary things called "lins" and "pins" submitted by a group of students.

(a) There are at least 2 distinct pins.

(b) A lin is a set of pins containing at least 2 distinct pins.

(c) Each set of 2 distinct pins is contained in exactly one lin.

(d) A lin does not contain all the pins.

These statements appear to be meaningless. However, they contain enough information to permit you to decide whether or not the statements below follow logically from them. Make these decisions and be prepared to support them.

1. There is at least one lin.
2. There are at least 3 pins.
3. There are exactly 3 pins.
4. There are exactly 2 lins.
5. There are at least 3 lins.
6. The intersection of 2 distinct lins contains at most 1 pin.

2-5. Incidence Postulates--Points and Lines.

In the last section we said that the incidence postulates are statements about geometric points, lines, and planes. They are statements which we feel are in agreement with our experiences in physical geometry. We do not prove these statements. We accept them as a starting point in our project of organizing a formal geometry.

To get started, then, we draw upon our experiences with physical points and lines. Perhaps the most common experience is that of drawing a line through two points using a straightedge or a ruler. On the basis of this and other experiences we find the following ideas reasonable: that a line is a set of points, that we can draw a line through any two points, and that no line contains all the points considered in geometry.

With these ideas in mind we proceed to state a definition and four postulates about geometric points and lines.

DEFINITION. Space is the set of all points.

Postulate 1. Space contains at least two distinct points.

Postulate 2. Every line is a set of points and contains at least two points.

Postulate 3. If P and Q are two distinct points, there is one and only one line that contains them.

Postulate 4. No line contains all points of space.

Does it seem strange that our second postulate says "at least two points"? Since we feel that a line contains a great many more than two points, it is natural to ask why we don't come right out and say so. This could be done. But we believe our development is more instructive since it permits us to see more clearly how some properties of points and lines depend upon other properties.

We consider now some other relationships between geometric points and lines which are suggested by our experiences in physical geometry. Certainly we would like to have lines in our geometric space. Also, if two distinct lines intersect, we would like them to intersect in exactly one point. Surely this is what we want on the basis of our experiences in physical geometry.

The interesting thing about these statements is that we don't need to postulate them. We can deduce them from our postulates and do so below. This will help you to see how these relationships depend upon the relationships stated in the postulates.

We now show that there is a line in our geometric space. Postulate 1 tells us that there are two distinct points. Postulate 3 tells us that there is a line which contains these two points. We know from Postulate 2 that a line is a set of points. Since all points are contained in space we conclude that this line is contained in space. Therefore we may write the following theorem.

THEOREM 2-1. Space contains at least one line.

Postulate 1 tells us that there are at least two points in space. Using Theorem 2-1 and our postulates we can deduce that there are at least three points, and moreover, that there are at least three points not all contained in any one line. The reasoning proceeds as follows.

From Theorem 2-1 we know that space contains at least one line. Postulate 2 tells us that this line contains at least two points. Postulate 3 tells us there is no other line which contains these two points. Postulate 4 implies that there is a third point not on this line and therefore distinct from the first two. Thus we have another theorem.

THEOREM 2-2. Space contains at least three distinct points not in one line.

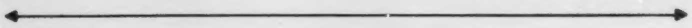
From now on most of our theorems will be deduced from a combination of preceding theorems and postulates, as is the case for Theorem 2-2. It appears that this theorem is deduced by using only a preceding theorem and Postulates 2, 3, and 4. Do you see how Postulate 1 is also involved in this deduction?

For convenience in the work we are about to do (as well as in later work) we introduce symbols to name points and lines, and figures to represent them.

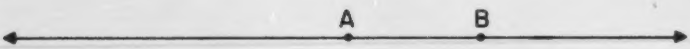
Notation. We shall denote the line determined by two points, A and B , by either of the symbols

$$\overleftrightarrow{AB} \quad \text{or} \quad \overleftrightarrow{BA} .$$

In our figures, we shall represent a line by the symbol



or by



if we wish to indicate that two particular points, A and B, are on it. Occasionally it will be more convenient to designate a line by a single letter (usually lower case) such as ℓ. If several lines are being considered, we may name each by a different letter, such as

$$\ell, m, n, \ldots$$

or we may use a single letter with distinguishing subscripts, such as

$$\ell_1, \ell_2, \ell_3, \ldots$$

We return to our discussion of relationships between geometric points and lines. Recall that we wanted points and lines in our formal geometry to have the following property: if two distinct lines intersect, they intersect in exactly one point. This is not one of our postulates. However, it can be deduced from them. But first, let us deduce that there are at least two lines in space.

Theorem 2-2 tells us that space contains at least three distinct points not on one line. Let A, B, and C denote three such points. Postulate 3 tells us that $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are lines. Since C cannot lie on $\overleftrightarrow{AB}$, and since $\overleftrightarrow{AB}$ is the only line containing A and B (Postulate 3), we see that $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are distinct. It follows that space contains at least two distinct lines. Do you see how the reasoning may be extended to deduce that space contains at least three distinct lines?

Suppose, now, that ℓ_1 and ℓ_2 are two distinct intersecting lines. Since they intersect, they have at least one point in common. Call one such point A. Either they have another point in common or they don't. Suppose that there is another point B which lies in both ℓ_1 and ℓ_2. Then Postulate 3 tells us that there is exactly one line containing A and B. This would mean that ℓ_1 and ℓ_2 are the same line. Since ℓ_1 and ℓ_2 are distinct, we have ruled out the possibility that they have a second point in common, and, we conclude that they have exactly one point in common. Thus we have deduced the following theorems.

THEOREM 2-3. Space contains at least two lines.

THEOREM 2-4. If two distinct lines intersect, they intersect in exactly one point.

Up to now we have four postulates and four theorems. You may feel that all eight of these statements are obvious and simple statements reflecting your intuitive feelings about lines and points in space. You may wonder why we don't postulate everything. As we said before, we postulate some statements and deduce others from them so that we can have a logical organization showing the dependence of one statement upon another. You may wonder also whether it is possible to start with a different set of postulates. Yes, it is. A postulate in one book may be a theorem in another book.

Sometimes (as in Theorem 2-2, for example) we wish to distinguish between a set of points which is contained in a line and a set which is not. To do this we introduce the terms collinear and noncollinear.

DEFINITION. The points of a set are collinear if and only if there is a line which contains all of them.

This definition permits us to say both

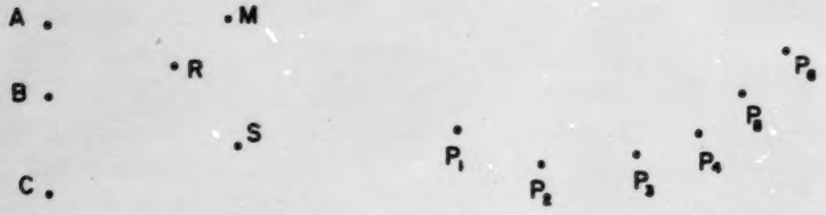
If the points of a set are collinear,
they are contained in one line

and

If the points of a set are contained
in a line, they are collinear.

Problem Set 2-5

1. The three sets $\{A,B,C\}$, $\{M,R,T\}$, $\{P_1,P_2,P_3,P_4,P_5,P_6\}$ are pictured below. Which of these appears to be a set of collinear points?



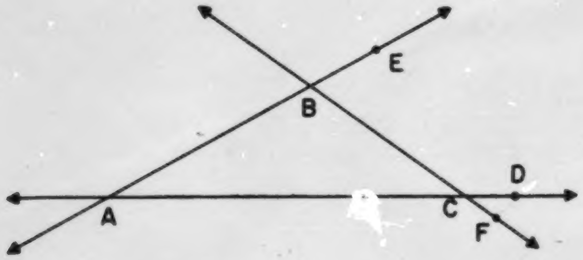
2. Consider each of the following statements. Which of them are true with respect to the postulates, definitions and theorems which have been developed thus far? Indicate a reason for each of your answers.

   (a) If a set consists of three distinct points then the elements of the set are collinear.

   (b) If a set consists of two distinct points then the elements of the set are collinear.

   (c) If a set consists of at least two distinct points then the elements of the set are collinear.

3. Consider the set consisting of the six points named in the diagram below, and no others.



   (a) Identify three subsets each containing three collinear points by listing the members.

(b) List the members of four subsets each containing four noncollinear points of which three are collinear.

(c) List the members of four subsets each containing four noncollinear points of which no three are collinear.

(d) Name all the lines which are not drawn in the sketch but which are determined by pairs of the points named in the sketch.

4. Explain in a paragraph how to justify the statement that space contains at least three lines.

5. Which postulates would enable you to deduce that given a point, there is a line which contains that point?

2-6. Incidence Postulates--Points, Lines and Planes.

We turn next to planes and their relations with points and lines. The points of formal geometry are suggested by dots and tips of pins; lines by stretched strings; planes by flat surfaces like pieces of stiff cardboard. Let us look for some key relations. Suppose we wish to rest a cardboard on two nails as shown in Figure 1. Will the cardboard assume a fixed position without sliding off?

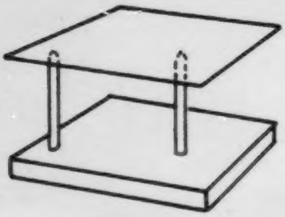


Figure 1

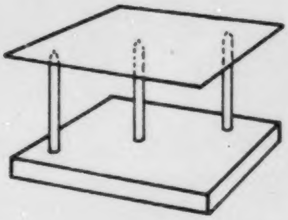


Figure 2

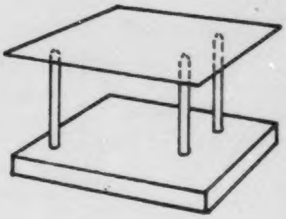


Figure 3

Let us take three or more nails whose tips are in a line. Will the cardboard now assume a fixed position? Now let us take three nails, not in a line. Will the cardboard assume a fixed position? Can we set still more nails that will not disturb this position? That would disturb it?

These experiments suggest that we add to our description of points, lines, and planes in formal geometry the following postulates.

Postulate 5. Every plane is a set of points and contains at least three noncollinear points.

Postulate 6. If P, Q, R are three distinct noncollinear points, then there is one and only one plane which contains them.

Postulate 7. No plane contains all points of space.

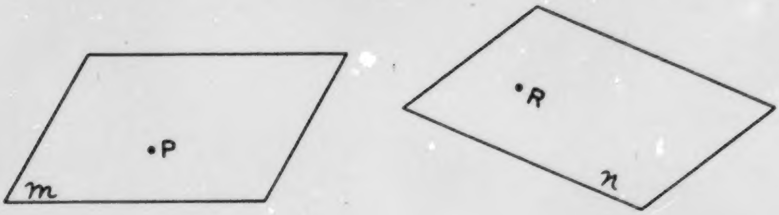
Certainly, we want our description to include a statement that there are planes. We will not state this as a postulate, since it can be deduced a theorem. Indeed, we can deduce that not only is there a plane, but that every point is in at least one plane. Suppose that P is a point; then, by Postulate 1, there is another point Q. From Postulate 3, these points, P and Q, determine the line $\overleftrightarrow{PQ}$. By Postulate 4, there is a point R not on $\overleftrightarrow{PQ}$. Thus R, P, Q are noncollinear points. Hence, by Postulate 6, there is a plane containing them. Thus, we have shown, that given a point P, there is a plane to which it belongs.

THEOREM 2-5. If P is a point, there is a plane that contains it.

Using Theorem 2-5, it is easy to deduce that there are at least two planes. From Postulate 1, there is a point P. By Theorem 2-5 there is a plane m that contains it. From Postulate 7, there is a point R not in m. By Theorem 2-5 there is a plane n that contains R. Planes m and n are different since R is not in m but is in n. Finally, we observe that since every plane is a set of points, every plane is contained in space. Thus we have deduced the following theorem.

THEOREM 2-6. Space contains at least two planes.

It is often convenient to make diagrams or drawings to accompany a deduction such as that for Theorem 2-6. In making such drawings, we shall represent a plane by a parallelogram as in the figure, remembering when we do, that the sides of the parallelogram are not the "edges" or "ends" of the plane.



Often, as here, a plane is named by using a single script capital letter, such as $\mathcal{M}$ or $\mathcal{N}$. Sometimes it is convenient to refer to the plane determined by three noncollinear points A, B, C, say, as "plane ABC."

From Theorem 2-2 we know there are three noncollinear points. By Postulate 6, there is just one plane that contains them. Now, by Postulate 7, there is a point not in this plane. Hence, we conclude that these four points are not contained in one plane. To distinguish between a set of points that is contained in a plane and one that is not, we introduce the terms coplanar and noncoplanar.

> **DEFINITION.** The points of a set are coplanar if and only if there is a plane which contains all of them.

Using this terminology our deduction is stated as a theorem.

THEOREM 2-7. Space contains at least four noncoplanar points.

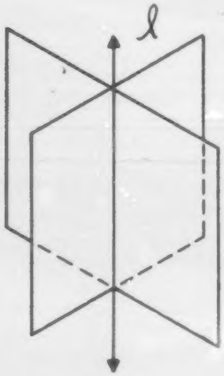
Notice that our seven postulates taken collectively have enabled us to deduce that space contains at least four points.

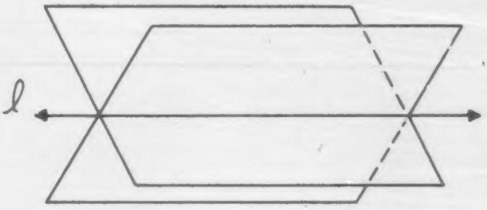
So far our postulates have not dealt directly with the relationships between lines and planes. One such relationship is suggested by the familiar process of checking the flatness of a surface with a straightedge or with a stretched piece of string. Another relationship is drawn from the familiar observation that two adjacent flat walls of a room intersect in a straight line, or that a folded piece of paper forms a straightedge. From these ideas we formulate the following statements as postulates.

Postulate 8. If two distinct points of a line belong to a plane, then every point of the line belongs to that plane.

Postulate 9. If two distinct planes intersect, then their intersection is a line.

The figures below may help you "see" two planes intersecting in a line.





Practice making such figures. Some people think the figure is clearer if the parts of lines "hidden" by portions of planes are shown with "dotted lines." (For a discussion on how to make drawings of configurations in space geometry, see Appendix V.)

Problem Set 2-6

1. Which postulate explains why, on a level floor, a four-legged table sometimes rocks while a tripod or a three-legged table is always steady?

2. If we know that three noncollinear points, M, N, R, lie in plane $\mathcal{F}$ and also lie in plane $\mathcal{S}$, what can we conclude about $\mathcal{F}$ and $\mathcal{S}$? Why?

3. Consider the following statements. In each case indicate whether you think the statement can be deduced from our first nine postulates. Justify your answer.
    (a) A set of points can be both in a plane and in a line.
    (b) If the points of a set are collinear, then they are also coplanar.

4. Write a short paragraph to show how our postulates and theorems can be used to prove:
    (a) If four distinct points are noncoplanar then they are noncollinear.
    (b) If four distinct points are noncoplanar then any three of these points are noncollinear.

    Hint: In (a) start by assuming that the four points are collinear and then try to make deductions which contradict the hypothesis.

5. In addition to the nine postulates of our formal geometry, assume, for this problem only, that space contains exactly four points, A, B, C and D. Justify your answers to the following questions.
    (a) May these 4 points be coplanar? Why?
    (b) May these 4 points be collinear? Why?
    (c) A plane then contains exactly how many points?
    (d) If A, B, and C are coplanar, may A, B and C be collinear? May A, B and D be collinear? Why?
    (e) A line, then, contains exactly how many points? Why?
    (f) Name all the lines that space contains.
    (g) Name all the planes that space contains.

2-7. Three Theorems.

Some relationships among points, lines, and planes are stated formally in the following theorems. Can you think of an experience that suggests them? In each case the proofs are outlined and you are supposed to tell which postulate or theorem justifies some of the steps. For each theorem draw one or more appropriate diagrams to help you in understanding the proof.

THEOREM 2-8. If a line intersects a plane not containing it, the intersection is a single point.

Proof:

By hypothesis, we have a line ℓ and a plane m

By hypothesis, ℓ intersects m .

By hypothesis, m does not contain ℓ .

There is a point P in the intersection of ℓ and m (because this is what intersects means).

There are two possibilities: either P is the only point in the intersection, or it isn't.

Suppose there is another point Q distinct from P in the intersection.

Then Q lies in ℓ .

Then ℓ must be contained in m by Postulate _____ . (Tell which one.)

Does this contradict our hypothesis?

Does this prove the theorem?

THEOREM 2-9. A line and a point not on that line are contained in exactly one plane.

Proof:

By hypothesis, we have a line ℓ and a point P .

By hypothesis, P does not lie in ℓ .

By Postulate _____ we know that ℓ contains two distinct points, A and B .

By Postulate _____ we know that A and B do not lie on any line except ℓ .

Then A, B, and P are noncollinear.

By Postulate _____ A, B, and P are contained in exactly one plane m .

By Postulate _____ $\overleftrightarrow{AB}$ is contained in m .

Then m contains ℓ and P .

Is there any other plane which contains A, B, and P ?

Is there any other plane which contains ℓ and P ?

Does this prove the theorem?

THEOREM 2-10. If two distinct lines have a point in common, there is exactly one plane which contains them.

Proof:

By hypothesis, there are two distinct lines ℓ_1 and ℓ_2 .

By hypothesis, ℓ_1 and ℓ_2 intersect.

Then we know by _______________ that ℓ_1 and ℓ_2 have exactly one point P in common.

Then _______________ tells us that there is a point A on ℓ_1 different from P .

Also _______________ tells us that there is a point B on ℓ_2 different from P .

By Theorem _____ there is exactly one plane containing ℓ_1 and B .

This plane contains P and B and therefore by _______________ contains ℓ_2 .

Is there a plane containing ℓ_1 and ℓ_2 ?

Is there exactly one plane containing ℓ_1 and ℓ_2 ?

Does this prove the theorem?

2-8. Summary.

In this chapter we discussed sets, one-to-one correspondences and began our formal development of geometry. In connection with the concept of sets, the following terms were introduced.

member	belongs to	subset	intersection
element	contains	union	empty set

The formal geometry developed involved the undefined terms, point, line, and plane, and the defined terms, space, collinear, and coplanar.

The postulates introduced can be classified as follows: first, a postulate assuring the existence of some points in space; second, three postulates relating points and lines; third, three analogous postulates relating points and planes; finally, two postulates relating lines and planes directly. We found that some statements about geometric points, lines and planes did not have to be postulated since they could be deduced from our earlier postulates. These statements we listed as theorems.

These postulates and theorems are listed in order at the end of the volume. The index gives the appropriate page reference for each formal definition; they are not listed separately.

Review Problems

1. Let $S_1 = \{*, \neq, \ \}$, $S_2 = \{*\}$, $S_3 = \{-, +, \div, X\}$:

   Give the elements of set S (denote the set by using brace notation) if S is the

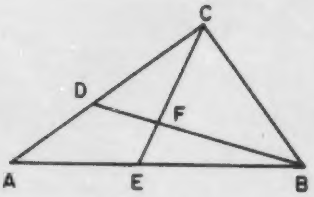
   (a) intersection of S_1 and S_2 .
   (b) union of S_1 and S_2 .
   (c) intersection of S_2 and S_3 .
   (d) union of S_2 and S_3 .
   (e) intersection of S_1 , S_2 , and S_3 .
   (f) union of S_1 , S_2 , and S_3 .

2. Given a set of points with two elements, how many lines are determined?

3. Complete the following statement.

   (a) The points of a set are collinear if and only if
   ____ .

   (b) Complete the following statements using the accompanying diagram.

   (1) Points D, C, _____ appear collinear.

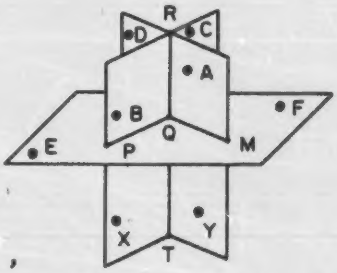
   (2) Points E, F, and _____ appear collinear.

   (3) Points B, _____ , and A appear collinear.

   (4) Points A, B, C, D, E, F appear __________ .

4. (a) The points of a set are coplanar if and only if
   ____ .

   (b) Using the accompanying diagram, complete the following statements.

   (1) Points E, F, _____, _____ and _____ appear coplanar.

   (2) Points C, B, X, _____ , _____ , _____ , and _____ appear coplanar.

   (3) Points D, Y, A, _____ , _____ , _____ , and _____ appear coplanar.

   (4) Points _____ , _____ , _____ , appear coplanar and collinear.

5. Let $P = \{2,3,4\}$ and $N = \{4,9,16\}$.

   (a) Show a one-to-one correspondence between P and N such that each element of P is matched with its square in N .

   (b) Show a one-to-one correspondence in which each element of N is greater than its corresponding element of P .

6. (a) May 4 points be collinear?
   (b) Must 2 points be collinear?
   (c) Must 4 points be coplanar?

7. How many different lines may contain

   (a) one given point?
   (b) a given set of two distinct points?

8. How many different planes may contain

   (a) one given point?
   (b) a given set of three points?
   (c) a given set of two distinct points?

9.

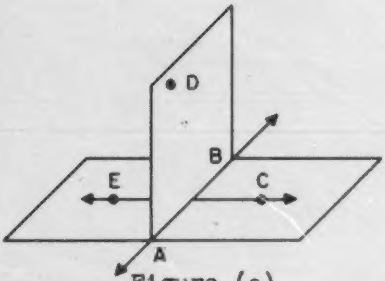


Figure (a)

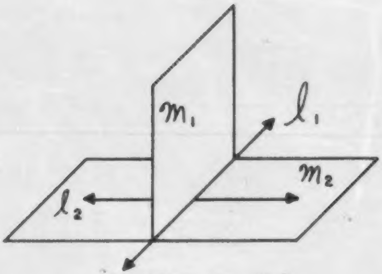


Figure (b)

Figure (b) is a copy of Figure (a), except for labeling. In the left-hand column are listed parts of Figure (a). Match these with parts of Figure (b) listed in the right-hand column.

Parts of Figure (a)	Parts of Figure (b)
(1) EC	(a) ℓ_1
(2) Plane ABC	(b) ℓ_2
(3) Plane ABD	(c) m_1
(4) Plane EBA	(d) m_2
(5) AB	
(6) The intersection of plane ABC and plane ABD .	

Does the second column suggest an advantage of the subscript way of labeling?

10. Given:

    (a) Points A, B, C lie in plane m .
    (b) Points A, B, C lie in plane n .

    Can you conclude that plane m is the same as plane n ? Explain.

11. List all the combinations of points and lines we have studied which determine a single plane.

12. Line ℓ_1 intersects plane $\mathcal{E}$ in P but does not lie in $\mathcal{E}$. Line ℓ_2 lies in plane $\mathcal{E}$ but does not contain point P . Is it possible for ℓ_1 and ℓ_2 to intersect? Explain.

13. Suppose ℓ is a line and m is a plane. Make a sketch for each of the following.

    (a) ℓ intersecting m but not contained in m .
    (b) The intersection of ℓ and m as an empty set.
    (c) ℓ intersecting m and contained in m .

14. Point P lies in both of the distinct planes m and n which intersect in line $\overleftrightarrow{AB}$. Would it be correct to say that P lies in $\overleftrightarrow{AB}$? Explain.

15. Let Q be the set of all points in the intersection of line ℓ_1 and line ℓ_2. Sketch a figure illustrating the situation if Q

(a) is the empty set.
(b) contains one element.
(c) is the same as ℓ_1.

16. Let A be the set of points in the intersection of plane R and plane S. Sketch a figure illustrating the planes if A

(a) is the empty set.
(b) contains at least two members.
(c) contains three noncollinear points.

17. Write each of the following statements in the "if-then" form.

(a) Zebras with polka dots are dangerous.
(b) Rectangles whose sides have equal lengths are squares.
(c) There will be a celebration if Oklahoma wins.
(d) A plane is determined by any two intersecting lines.
(e) Cocker spaniel dogs are sweet tempered.
(f) Two distinct lines have at most one point in common.
(g) Every geometry student knows how to add integers.
(h) A line and a point not on the line are contained in exactly one plane.

18. Indicate which part of each of the following statements is the hypothesis and which part is the conclusion. If necessary, rewrite in "if-then" form first.

(a) If John is ill, he should see a doctor.
(b) A person with red hair is nice to know.
(c) Four points are collinear if they lie on one line.
(d) If I do my homework well, I will get a good grade.
(e) If a set of points lies in one plane, the points are coplanar.

Chapter 3

DISTANCE AND COORDINATE SYSTEMS

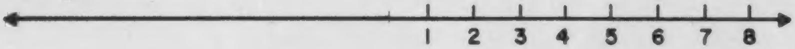
3-1. Introduction.

The postulates which we introduced in the preceding chapter and the theorems which we proved discussed the interrelations of points, lines, and planes thought of simply as sets of points. No ideas of size or shape, that is, no questions of measurement, were involved in any way.

Our next major objective is to introduce the concept of distance, together with related topics. As usual, our procedure will be to adopt some additional postulates which will make precise our intuitive notions and then to explore the consequences of these postulates. However, since distances are measured in terms of numbers, we first examine various properties of the real numbers, some of which you learned in algebra. In later chapters we shall use real numbers in measuring angles, areas of surfaces, and volumes of solids.

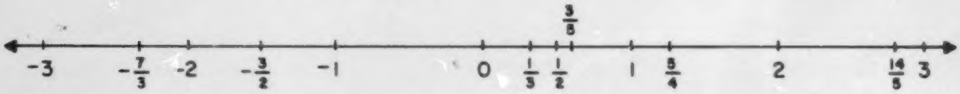
3-2. The Set of Real Numbers.

The system of real numbers that you used in algebra and will use this year in studying measurement is the result of several stages of development. At an early age you learned about the counting numbers (sometimes called natural numbers, or positive integers) 1, 2, 3, 4, 5, 6, ... As the three dots suggest, the list of counting numbers continues without end; that is, there is no "last" or "largest" natural number. We often think of the positive integers arranged on a line, in an unending fashion toward the right.



The positive integers, together with the integer 0 (which is neither positive nor negative), and the negative integers -1, -2, -3, ... form the set of all integers. The introduction of fractions leads to the set of rational numbers. Numbers like $\frac{1}{2}$, $\frac{5}{4}$, $\frac{-7}{9}$, $\frac{-41}{3}$, 1.3, -8.72 are rational numbers, because each of them is the quotient of two integers. Every integer is also a rational number; in other words, the set of integers is a subset of the set of rational numbers.

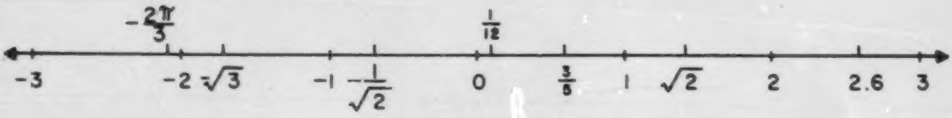
The rational numbers can be arranged on the number-line.



However, only a few of the rational numbers represented in a diagram can be labeled, because the rational numbers are so closely packed together on the number-line. For example, between $\frac{1}{2}$ and $\frac{3}{5}$, there are many other numbers, such as 0.55, 0.58, 0.501, 0.52837, each of which is rational. Between any two rational numbers there are infinitely many other rational numbers, including their average. However, the rational numbers, in spite of their density on the number-line, do not "fill up" the line.

In order to "fill the gaps," we extend our number system again. We introduce the so-called irrational numbers. One example is $\sqrt{2}$; another is the number π, which appears in the familiar formula $A = \pi r^2$ for the area enclosed by a circle. All the numbers we have mentioned, the rational and the irrational numbers, form the set of real numbers. The set of rational numbers is a subset of the set of real numbers. It is sometimes difficult to determine whether a particular real number belongs to the subset of rational numbers or not. The case of the number $\sqrt{2}$, however, is relatively easy and is discussed in Appendix IV. Although we often, in computational work, replace $\sqrt{2}$ by a rational number which is nearly equal to it, such as 1.414, we must not conclude that $\sqrt{2}$ is itself a rational number.

Each real number may be represented on the number-line. Recalling that π is approximately 3.14, would you expect π to be a little toward the right of 3 ? Since $\sqrt{2}$ is approximately 1.4, where is $\sqrt{2}$ located? Several real numbers are represented in the following diagram.

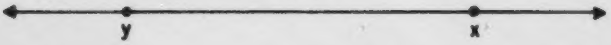


Whereas your study of algebra may have emphasized the integers and rational numbers, our work in geometry will rely heavily on the set of real numbers. The representation of this set by means of the points on a line will be extremely helpful. In fact, we shall soon introduce the concept of a coordinate system on a line, a notion which in formal geometry plays a role somewhat resembling the idea of a number-line in informal geometry.

Your previous mathematical training has taught you a good deal about the behavior of real numbers under the fundamental operations of addition, subtraction, multiplication, and division. Although you may have experimented and discovered these properties by inductive reasoning, they can be deduced from a few simple statements. These statements are among the postulates of algebra, for algebra is based on initial assumptions in the same way as geometry. We will not interrupt our study of geometry to develop these properties of the real numbers but will merely borrow them from algebra. Appendix II gives the postulates which you need to deduce these properties, and you may find it challenging to examine them and to study sample proofs based on them.

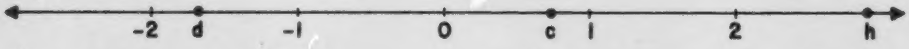
However, some properties of the real number system which will be of importance to us may be unfamiliar to you. We pause briefly to discuss inequalities.

An important characteristic of the real numbers is that we can compare them. Of two distinct real numbers, one of them is "larger than," or greater than, the other. Pictorially speaking, we compare two numbers by noting the relative positions of the corresponding points on the number line. Each of the two numbers is represented by a point; the point on the right of the other (as we have drawn the number-line) represents the larger of the two numbers.



In the diagram, the number x is greater than the number y.

A shorthand expression for "x is greater than y" is: $x > y$.



Some examples involving numbers labeled in the above diagram are: $2 > 1$, and $1 > -2$, and $c > 0$, and $c > d$, and $0 > -2$, and $h > 2$. Is 1 greater than $-1,000,000$?

We can compare two numbers, say x and y, by noting whether their difference, $x - y$, is positive or negative. For instance, $4 - (-\frac{3}{2})$ is positive; and 4 is greater than $-\frac{3}{2}$. To check that $-1 > -10$, we observe that the difference $(-1) - (-10)$, namely 9, is positive. In general, given a pair of real numbers, the first is greater than the second if and only if the difference obtained by subtracting the second from the first is a positive number.

Problem Set 3-2

1. Consider the following six sets:

   A is the set of real numbers
   B is the set of integers
   C is the set of rational numbers
   D is the set of irrational numbers
   E is the set of natural numbers
   F is the set of positive integers

   (a) Are any of the above sets equal?
   (b) Which of the above sets are subsets of A ? of B ? of C ?
   (c) Which set is the intersection of B and C ?
   (d) Which set is the union of C and D ?
   (e) Which set is the union of B and F ?
   (f) What is the intersection of C and D ?

2. Using (if possible) the symbol $>$, compare the numbers in each of the following pairs. Which is to the right of the other on the number-line?

(a) $2, \ -5$

(b) $-3, \ -7$

(c) $\sqrt{5}, \ \sqrt{3}$

(d) $-\frac{3}{2}, \ 0$

(e) $-\frac{5}{2}, \ -\frac{10}{4}$

(f) $\frac{16}{3}, \ \frac{21}{4}$

(g) $-\sqrt{\frac{16}{9}}, \ -\sqrt{\frac{25}{16}}$

3. Is it true for all integers a, b, c that if $a = b + c$, then $a > b$? If it is true, explain why. If it is false, suggest a change in the statement so that the new statement will be true. Is it useful to consider $a - b$ in forming your answer?

4. For each pair of numbers (x,y) in the table, compute the difference, $x - y$, and tell which number is the larger by using the symbol $>$, as illustrated in (a).

	x	y	x - y	Inequality
(a)	5	2	3	$5 > 2$
(b)	-4	-6		
(c)	-12	-2		
(d)	8	-3		
(e)	-4	2		
(f)	$\frac{1}{2}$	$\frac{1}{3}$		
(g)	-3	3		
(h)	a $a > 0$	b $0 > b$		

*5. Compare the numbers in the following pairs by using the symbol, $>$.

(a) $1, \ -3$

(b) $1 + 5, \ -3 + 5$

(c) $1 - 5, \ -3 - 5$

(d) $-6, \ -2$

(e) $-6 + 3, \ -2 + 3$

(f) $-6 - 3, \ -2 - 3$

*6. Tell whether each of the following is true or false when x is replaced by (1) a positive number; (2) a negative number.

(a) $5 + x > 2 + x$ (d) $4 + x > -4 + x$
(b) $-6 + x > -12 + x$ (e) $-20 + x > -21 + x$
(c) $-8 + x > 0 + x$ (f) $1000 + x > 3000 + x$

*7. Compare the numbers in the following pairs by using the symbol $>$.

(a) (i) $5, 2$
(ii) $3 \cdot 5,\ 3 \cdot 2$
(iii) $(-3) \cdot 5,\ (-3) \cdot 2$

(b) (i) $2,\ -6$
(ii) $3 \cdot 2,\ 3(-6)$
(iii) $(-3) \cdot 2,\ (-3)(-6)$

(c) (i) $12, 8$
(ii) $\frac{1}{2} \cdot 12,\ \frac{1}{2} \cdot 8$
(iii) $(-\frac{1}{2}) \cdot 12,\ (-\frac{1}{2}) \cdot 8$

*8. Tell whether each of the following is true or false if x is replaced by (1) a positive number; (2) a negative number.

(a) $5x > 2x$ (d) $-5x > -2x$
(b) $-6x > -5x$ (e) $6x > -5x$
(c) $-5x > 2x$ (f) $-6x > 5x$

*9. Test whether the sentence, "If $x > y$, then $ax > ay$," is true or false in each of the following:

(a) (1) $x = 3,\ y = 2,\ a = 4$
(2) $x = 3,\ y = 2,\ a = -4$

(b) (1) $x = 2,\ y = -3,\ a = 4$
(2) $x = 2,\ y = -3,\ a = -4$

(c) (1) $x = -2,\ y = -5,\ a = 3$
(2) $x = -2,\ y = -5,\ a = -3$

3-3. The Ordering of the Real Numbers.

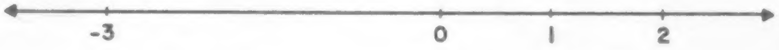
In a deductive development of algebra, the notion of "greater than," discussed at the end of the preceding section, is made more precise by a set of postulates. From these postulates, or basic assumptions, several additional useful properties may be deduced as theorems. A summary of this development appears in Appendix III.

In our study of the greater-than relation, we shall simply examine further the various properties which we need and then collect together their statements for ready reference.

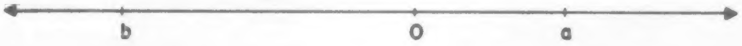
First, we note that any two distinct real numbers can be compared: one of them is greater than the other. For the numbers 5 and $\frac{3}{7}$, we can say that $5 > \frac{3}{7}$; considering -4 and 1.2, we observe that $1.2 > -4$. In general, if x and y are distinct numbers, then either $x > y$, or else $y > x$.

An important special case involves the comparison of a real number b and the number 0. The statement that $b > 0$ is often expressed in words by saying that b is positive. On the other hand, the statement that $0 > b$ is expressed by saying that b is negative. Any real number r either is zero or else is positive or else is negative. In symbols, either $r = 0$ or $r > 0$ or $0 > r$.

Next, we note the transitive property. As a numerical example,



$2 > 1$ and $1 > -3$; moreover, $2 > -3$. For another illustration, if we know that $a > 0$ and $0 > b$, then we can conclude that $a > b$. Let us interpret this situation on the number-line. The hypotheses that $a > 0$ and $0 > b$ mean that a is to the right of 0 and that 0 is to the right of b.

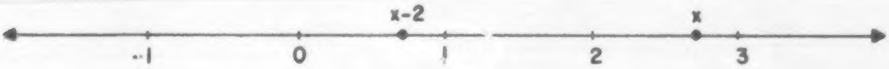


Thus a is to the right of b.

We have been illustrating transitivity. This property states that if x, y, z are real numbers such that $x > y$ and $y > z$, then $x > z$.

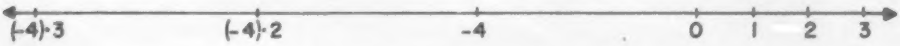
As an application of this property, if a number w is known to satisfy the condition that $w > 5$, we can conclude that $w > 0$. Explain why this conclusion is an application of the transitive property.

Our remaining properties discuss the relationship between an inequality and the arithmetic operations of addition and multiplication. Refer to Problems 5 and 6 in Problem Set 3-2. In Problems 5(a) and 5(b), we observed that $1 > -3$ and $1 + 5 > (-3) + 5$. As another example, if we add -2 to each number in the inequality $x > 2$, we obtain $x + (-2) > 2 + (-2)$, or more simply, $x - 2 > 0$.



These examples illustrate that an inequality is preserved if the same number is added to each number in the inequality. In symbols, if $x > y$, then $x + a > y + a$. In a like fashion, if the same number is subtracted from each number in an inequality, the inequality is preserved. If $x > y$, then $x - a > y - a$.

Now, let us try to discover the principle concerning multiplication. If $x > y$, can we conclude that $ax > ay$? Look again at Problems 7, 8, and 9 in Problem Set 3-2. Some parts of these problems suggest that the answer is "Yes," but other parts indicate a different conclusion in case the number a is negative. In particular, in Problem 9 you tested the case $x = 3$, $y = 2$, $a = -4$. The information that $3 > 2$ and that -4 is a negative number does not lead us to the conclusion that $(-4) \cdot 3 > (-4) \cdot 2$, but instead to the conclusion that $(-4) \cdot 2 > (-4) \cdot 3$.



Our basic principle concerning multiplication and inequalities becomes the following. Let x, y, a be real numbers such that $x > y$. If a is positive, then $ax > ay$; but if a is negative, then $ay > ax$.

We now summarize the algebraic properties of the order relationship. Afterwards we shall apply the properties in several examples.

Properties of Order

1. (Linear property) If x and y are two given distinct real numbers, then either $x > y$ or $y > x$, but not both.
2. (Definition of positive) A real number p is positive if and only if $p > 0$.
3. (Definition of negative) A real number n is negative if and only if $0 > n$.
4. (Transitive property) If x, y, z are real numbers such that $x > y$ and $y > z$, then $x > z$.
5. (Additive property) If x, y, a are real numbers such that $x > y$, then $x + a > y + a$.
6. (Multiplicative property) Let x, y, a be real numbers such that $x > y$. If a is positive, then $ax > ay$; if a is negative, then $ay > ax$; if $a = 0$, then $ax = ay$.

Example 1. If $x > 3$, show that $x - 3$ is positive.

Solution: $x > 3$; we have stated the hypothesis.

$x + (-3) > 3 + (-3)$; we have applied the additive property of order.

$x - 3 > 0$; we have used simpler names.

$x - 3$ is positive; we have applied the definition of positive.

Example 2. If $2x + 5$ is positive; show that $x > -\frac{5}{2}$.

Solution: $2x + 5$ is a positive number, by hypothesis.

$2x + 5 > 0$; by definition of positive.

$2x > -5$; by application of the additive property of order.

$x > -\frac{5}{2}$; give the reason.

Example 3. If $-4x > b$, show that $-\frac{b}{4} > x$.

Solution: $-4x > b$, by hypothesis.

$(-\frac{1}{4})b > (-\frac{1}{4})(-4x)$; we have applied the multiplicative property of order, noting that $-\frac{1}{4}$ is a negative number.

$-\frac{b}{4} > x$; we have used simpler names.

Problem Set 3-3a

1. State the property or properties of order which each of the following sentences illustrate.

   (a) If $x > -2$, then $x + 2 > 0$.

   (b) If $y > 3$, then $y - 4 > -1$.

   (c) If $-3 > -4$ and $-4 > -5$, then $-3 > -5$.

   (d) If $a > 4$, then $a > 0$.

   (e) If $10 > 3$, $3 > 0$ and $0 > -2$, then $10 > -2$.

   (f) If $\frac{1}{2} > \frac{1}{3}$, then $4\frac{1}{2} > 4\frac{1}{3}$.

   (g) If $4\frac{1}{2} > 4\frac{1}{3}$, then $\frac{1}{2} > \frac{1}{3}$.

   (h) If $-5x > 8$, then $-\frac{8}{5} > x$.

   (i) If $3x - 7 > 0$, then $x > \frac{7}{3}$.

   (j) If $x > y$, then $5x + 2 > 5y + 2$.

2. The transitive property may be extended to include any number of real numbers. Show that if x, y, z, w are real numbers, such that $x > y$, $y > z$, $z > w$, then $x > w$.

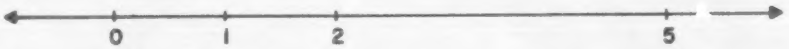
3. If $2x - 3$ is positive, show that $x > \frac{3}{2}$.

4. If $3 - 2x$ is positive, show that $\frac{3}{2} > x$.

5. If $5 - 10x$ is negative, show that $x > \frac{1}{2}$.

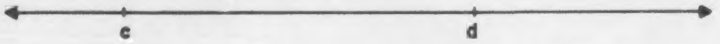
6. If $y > 3$, show that $5 > 3 - y$.

Thus far we have considered statements that one number is greater than another. This terminology seems to emphasize the larger of the two numbers. Sometimes we prefer to emphasize the smaller.



Instead of saying that 5 is greater than 2, we may say that 2 is less than 5. The symbol for "is less than" is $<$. We write $2 < 5$. The statement that -7 is less than $\frac{9}{4}$ may be written: $-7 < \frac{9}{4}$.

If c and d are real numbers, then the two statements, $d > c$ and $c < d$, have the same meaning.

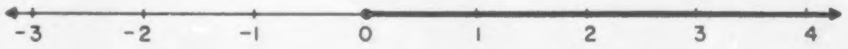


On the number-line, the statement that d is to the right of c means the same as the statement that c is to the left of d.

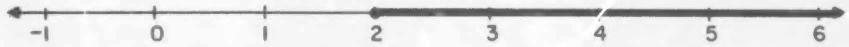
Since a statement involving the symbol $>$ can be rephrased using the symbol $<$, the properties of order can be expressed with the symbol $<$. We shall refer to each of the properties by the same name, no matter whether it is stated with one symbol or the other.

Often in mathematics we are concerned with real numbers which are not negative. The statement that a number d is not negative means that d must either be zero or be positive. In symbols, either $d = 0$ or $d > 0$. We abbreviate this sentence with the use of the symbol $\geq$. We write $d \geq 0$ in place of "either $d = 0$ or $d > 0$".

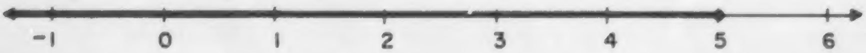
The set of all numbers d such that $d \geq 0$ is the same as the set of numbers which are either positive or zero. It is also the same as the set of real numbers which are nonnegative. On the number-line, the set of all numbers d such that $d \geq 0$ is represented by the set of points which consists of the point 0 and all points to the right of 0.



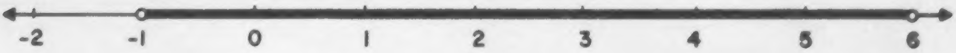
Suppose now that d and c are any two real numbers. The notation $d \geq c$ means that either $d = c$ or else $d > c$. On the number line the set of all numbers d such that $d \geq 2$ is represented by the set which consists of the point 2 and all points to the right of 2.



Instead of the notation $d \geq c$, we may also use $c \leq d$, which means that either $c = d$ or else $c < d$. On the number-line the set of all numbers c such that $c \leq 5$ is represented as in the following diagram.



If a, b, c are distinct real numbers, then the statement that "both $a < b$ and $b < c$" is often shortened by the notation $a < b < c$. On the number-line the set of all numbers b such that $-1 < b < 6$ is represented by the set of points which are both to the right of -1 and to the left of 6, that is, the set of all points which are between the point -1 and the point 6. Notice in the diagram the use of the small circles to indicate that neither of the points -1 nor 6 is included in the set.



Problem Set 3-3b

1. Restate the following in words:

   (a) $a < b$
   (b) $x > y$
   (c) $m \geq 3$
   (d) $n \leq 3$
   (e) $0 < 1 < 2$
   (f) $x > 0$
   (g) $-5 \leq x$ and $x \leq 5$
   (h) $-5 \leq x \leq 5$

2. Write as an inequality:

   (a) k is a positive number
   (b) r is a negative number
   (c) t is a number which is not positive
   (d) s is a nonnegative number
   (e) g is a number between 2 and 3
   (f) w is a number between 2 and 3 inclusive
   (g) w is a number between a and b, where $a < b$.

3. Express each of the following as a pair of inequalities.

   (a) $-2 < x < 5$ — How many integers are there in its solution set?

   (b) $3 \leq n < 11$ — How many odd integers are there in its solution set?

   (c) $0 \leq y \leq 7$ — How many integers are there in its solution set?

4. Given that x is a real number, plot each of the following sets on a number-line.

   (a) $x \geq 0$

   (b) $x < 0$

   (c) The union of the set of numbers x such that $x \geq 0$ and the set of numbers x such that $x < 0$.

   (d) The intersection of the set of numbers x such that $x \geq 3$ and the set of numbers x such that $x \leq 3$.

   (e) $2 \leq x < 7$.

5. (a) Describe the set of all z which satisfy the inequality $15z > 0$. What order property justifies your answer?

   (b) If $x > y$ and $x > z$ is it possible to tell which of the numbers y and z is the larger? Why?

6. Let m, x, p be real numbers. Each problem below contains two statements. If the second follows from the first by a property or properties of order, name the property or properties.

   (a) $3m > 2$; $-3m < -2$

   (b) $3x > 2 + x$; $2x > 2$

   (c) $5m > 10$; $m > 2$

   (d) $2x + 7 < x + 10$; $x < 3$

   (e) $-x < 3$; $x > -3$

   (f) $\frac{1}{p} < 2$ and $p > 0$; $1 < 2p$

   (g) $\frac{1}{m} < 0$ and $m < 0$; $1 > 0$

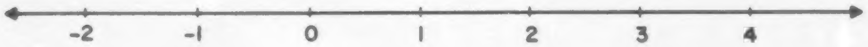
7. Some of the statements of the properties of order hold if $\geq$ replaces $>$; some do not. The following exercises involve some of the possibilities. State in each case whether or not the second statement follows from the first by the properties of order and equality. If it does not, explain why.

(a) $x + 5 \geq 7$; $x \geq 2$
(b) $a \geq 6$; $3a \geq 18$
(c) $9x \geq 15$; $x > \frac{5}{3}$
(d) $5x - 4 < 3x + 7$; $2x \leq 11$
(e) $4 < x + 3 \leq 9$; $1 < x \leq 6$
(f) $5 \leq -x < 10$; $-5 \geq x > -10$

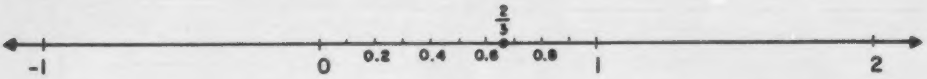
3-4. Distance.

In Chapter 2 we began our development of formal geometry. Informal geometry, which treats "what" and "how," has been contrasted with formal geometry, which stresses "why." Both are derived from our experiences in the physical world. In order to reason successfully, we need to understand the meanings of words precisely. Hence, familiar words, which sometimes have vague meanings in the real world, have been given precise definitions. In order to get started in our reasoning, we need to begin with certain agreements. Therefore, we have selected from our experience in the physical world a few statements which seem natural and we have adopted these, expressed in precise language, as postulates. With this foundation we have been able to deduce some theorems.

In Chapter 2, we formulated the incidence postulates, describing in formal geometry relationships among points, lines, planes, and space. In the present chapter we wish to introduce into our formal geometry some other ideas suggested by our experiences: ideas about a physical ruler and about the number-line.

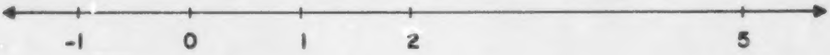


A number-line has become for us a convenient method of marking numbers and picturing the relationships between them. What sort of relationships between numbers are easily shown by the number line? One answer is the order relation: which of two given numbers is greater than the other? Seeing that 1 is to the right of -2 fits with the algebraic statement that $1 > -2$. Another possible answer is the betweenness relation.



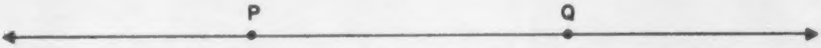
The number-line shows us that the number $\frac{2}{3}$ is between 0.6 and 0.7.

Does the number-line show other types of relationships? An example of another type is the following.



The number-line shows that 5 is farther from 1 than it is from 2. This answer suggests that the number-line pictures not only the order relation for numbers, but also shows how far apart numbers are. Now "how far apart" for numbers is determined by the operation of subtraction: $5 - 1 = 4$ tells us that the difference between 5 and 1 is 4.

But what does "how far apart" for points mean?



If P and Q are two distinct points, what do we mean by inquiring how far apart they are? In the real world, we answer such a question by using a physical ruler.

How can a physical ruler, graduated, say, in inches, be made? Clearly we must know what one inch is. We must have two points which are known to be "one inch apart." To assure that our physical ruler is accurate, the points we select should fit a standard. By law, the inch is defined in terms of the international unit for distance known as the meter. For many years, the meter was officially described by a pair of marks on a platinum-iridium bar kept under ideal conditions in France.

Although technological progress now demands a better standard in the physical world, the idea that two points provide a standard for measuring distances is important for our geometry.

Suppose any two distinct points in space are chosen. We could agree that this pair of points provides a unit. Thus we would say that the distance between these two chosen points is 1, and we could then measure the distance between any two points. If the original pair of points, say A and B, were one "inch" apart, then the distance between two other points, say P and Q, might be called "eight inches." The number 8 would be the measure of how far apart P and Q are, relative to the chosen "inch," that is, relative to the chosen pair {A,B} of points. The number, which is the measure of distance between two points, is, of course, a real number and not negative.

We are now ready to express, in a precise manner, some of the ideas concerning distance which we want in our formal geometry.

> Postulate 10. If A and A' are distinct points, there exists a correspondence which associates with each pair of distinct points in space a unique positive number such that the number assigned to the given pair of points {A,A'} is one.

DEFINITION. The set consisting of the two points A, A' mentioned in Postulate 10 is called the unit-pair.

DEFINITION. The number which corresponds, by Postulate 10, to a pair of distinct points is called the measure of the distance between the points, relative to the unit-pair {A,A'}.

DEFINITION. The measure of the distance between any point and itself, relative to {A,A'}, is the number zero.

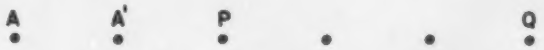
Suppose the points A and A' of a unit-pair are one inch apart. If two points P and Q are three inches apart, then the definition tells us that the measure of the distance between P and Q, relative to {A,A'}, is the number 3. Frequently, instead of the phrase, "the measure of the distance between two points," we simply say, "the distance between two points." Thus the distance between the points P and Q mentioned above is 3. That is, the distance is 3, if it is understood that {A,A'} is the unit-pair.

If a different unit-pair, say {B,B'}, replaced {A,A'}, then the distance between the same points P and Q very likely would be another number. For example, if B and B' were one "foot" apart, then the distance between P and Q would be the number $\frac{1}{4}$. In a discussion, if the unit-pair is understood, we often will not bother to mention it. However, if there is any chance of confusion, we must specify what unit-pair has been chosen.

> _Notation_. Let {A,A'} be a unit-pair. The distance between a point P and a point Q, relative to {A,A'}, is often denoted by: PQ.
> If it is desirable to identify the unit-pair, we use the notation: PQ (relative to {A,A'}).
> In the real world, the phrase "relative to {A,A'}" is often replaced by a physical unit, such as "inch" or "meter"; thus we use a phrase such as: PQ meters.

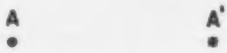
In all these cases, the symbol PQ names a number.

Example 1.



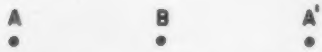
PQ (relative to {A,A'}) = 3.

Example 2.



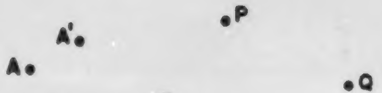
AA' = 1. (Note that here we have assumed that the unit-pair is {A,A'}.)

Example 3.



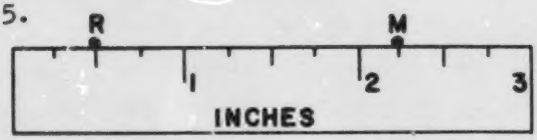
AB = $\frac{1}{2}$, and AA = 0.

Example 4.



$PQ = \frac{5}{2}$, and $QP = \frac{5}{2}$.

Example 5.



MR inches = $\frac{7}{4}$ inches.

In daily life we very seldom measure distances by referring directly to the legal standard inch. Instead we use physical rulers which provide copies of the standard. We consider the distances we measure with a high-quality copy to be the same as the distances we would obtain with the standard. This idea suggests the following postulate in our formal geometry.

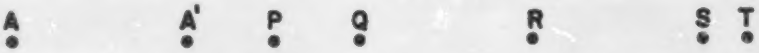
Postulate 11. If {A,A'} is any unit-pair and if B and B' are two points such that

BB' (relative to {A,A'}) = 1,

then for any pair of points, the distance between them relative to the unit-pair {B,B'} is the same as the distance between them relative to {A,A'}.

Problem Set 3-4

1. In this problem you are expected to use a ruler (graduated in inches) to measure distances between the points shown below.



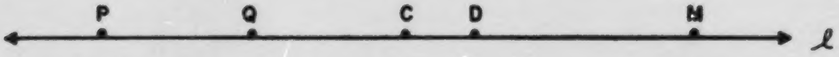
(a) Relative to {A,A'}, find A'Q, PQ, PR, PS, PT, ST, RR.

(b) Relative to {Q,R} , find A'Q, PQ, PR, PS, PT, ST, RR.

(c) What is QR (relative to {A,A'})? How do distances in (a) compare with the corresponding distances in (b)? What postulate does this illustrate?

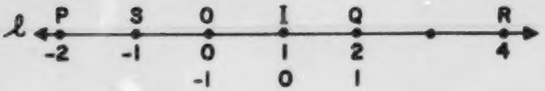
(d) Find RP
  (1) (relative to {A,A'}).
  (2) (relative to {Q,A'}).
  (3) (relative to {A,S}).
  (4) (relative to {S,T}).
  (5) (relative to {P,Q}).
  (6) (relative to {P,R}).

*2. In the figure, suppose that the part of ℓ shown between P and M is on the edge of a ruler.



(a) On this ruler if 0 were assigned to P, and 1 to Q, what numbers would you assign to PC, PD, PM ? In labeling the points of the ruler with numbers, what numbers would you assign to C, D, and M ?

(b) On this ruler if 0 were assigned to M, and 1 to D, what numbers would you assign to PM, QM, CM ? In labeling the points of the ruler with numbers, what numbers would you now assign to C, Q, P ?

*3.


(first correspondence)
(second correspondence)

On line ℓ the points have been assigned numbers (in ruler fashion) as indicated by the first correspondence on the diagram.

Suppose the ruler is moved to the right so that there is a new (or second) correspondence between the set of numbers and the set of points on ℓ , assigning 0 to I and 1 to Q.

(a) Copy the diagrams, and write on your copy the numbers which the second correspondence assigns to each of the other labeled points.

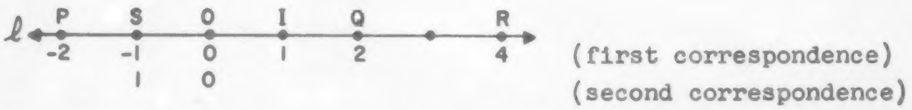
(b) Using the fact that the first correspondence assigned numbers in ruler fashion, compute the distances PQ, SR, PR, RO, QR (relative to {O,I}).

(c) What is IQ (relative to {O,I}) ?

(d) Without using the second correspondence, give the distances PQ, SR, PR, RO, QR (relative to {I,Q}). What postulate did you use?

(e) Using the fact that the second correspondence assigned numbers to the points in ruler fashion, compute the distances indicated in part (d). How do these computed distances compare with the corresponding distances given in (d) ?

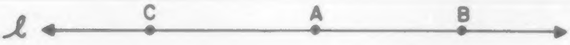
*4. On line ℓ again assign numbers to points in a ruler-like correspondence as indicated. Now consider the ruler turned so the scale extends in the "opposite direction" and associates 0 with point O and 1 with point S.



(a) Complete the number assignment of the second correspondence .

(b) Compute PR relative to the unit-pair in each correspondence. How do these distances compare? Why would you expect this?

*5.



Consider a line ℓ containing three points C, A, B, as indicated.

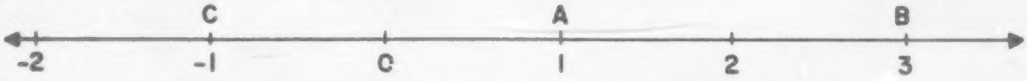
(a) Can you arrange a ruler with a scale so constructed that it could be placed along the line with its 0 corresponding to A and 8 to B ? If your answer is yes, describe where on the line the point corresponding to 1 would be located. Is this a unique point?

(b) Can you also arrange a ruler with a scale constructed so that it could be placed along the line with its 0 corresponding to A and 5 to C ? If your answer is yes, describe where on the line the point corresponding to 1 would be located. Is this a unique point?

3-5. Coordinate System on a Line.

In the preceding section we tried to analyze the number-line. We observed that the number-line helped us to picture "how far apart." This latter idea led us to formulate, in a precise manner, the notion of distance. Now we are ready to re-examine the number-line and attempt to capture its important features so that we may use these ideas in our formal geometry. The key feature is that the different points on the line are matched with various numbers. The correspondence between the two sets is one-to-one. But another feature of primary usefulness is the fact that this one-to-one correspondence enables us to compute the distance between points by subtracting numbers.

For example, consider a number-line marked in inches.



The distance between A and B (in inches) is 2; the number 2 can be computed by subtracting: namely, the numbers associated with B and A are 3 and 1, and $3 - 1 = 2$. The distance between B and C (in inches) is 4; the numbers corresponding to B and C are 3 and -1, and their difference $3 - (-1) = 4$. If the number -15 and the point D are matched on the above number-line, then $CD = (-1) - (-15) = 14$.

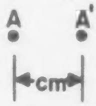
The concept in coordinate geometry which corresponds to the number-line in informal geometry is the general idea of a coordinate system. By properly associating points with numbers, we acquire a powerful tool whose importance will gradually unfold during the year's work. We now introduce a coordinate system on a line. This development, besides giving us important consequences about geometry on a line, will serve as a basis in later chapters for discussing coordinate systems in a plane or in space.

DEFINITION. Let $\{A,A'\}$ be any unit-pair and let ℓ be any line. A coordinate system on ℓ relative to the unit-pair $\{A,A'\}$ is a one-to-one correspondence between the set of all points on ℓ and the set of all real numbers with the following property: If numbers r and s correspond to points R and S on ℓ and if $r > s$, then $r - s$ is the same as the number RS (relative to $\{A,A'\}$).

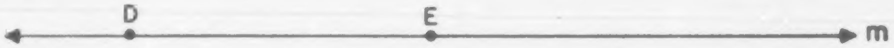
DEFINITIONS. The origin of a coordinate system on a line is the point which corresponds to the number 0. The unit-point of a coordinate system on a line is the point which corresponds to the number 1.
The number which a given coordinate system on a line assigns to a point R on the line is called the coordinate of the point R in the coordinate system.

These definitions describe what a coordinate system is and some of the words we find useful in talking about a coordinate system. However, there is an important issue which we should not overlook. A coordinate system has been described as a correspondence between two sets with certain very specific properties. How do we know, in our formal geometry, that such a correspondence is possible? If such a correspondence is possible, is there more than one coordinate system on a line? What, if anything, can we say about how many of these correspondences there are? In order to answer these questions, we once again return to our experiences with the number-line and use them to formulate a postulate which will give us the answers we need.

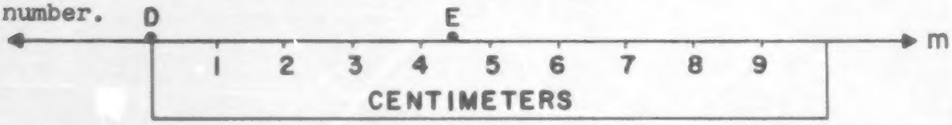
Suppose we choose two points, A and A', one centimeter apart, as a unit-pair.



Suppose we have a line m in the real world and two distinct points D and E on m.

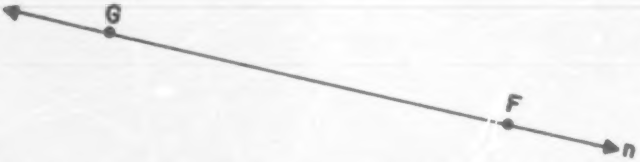


According to our experience with a physical ruler, we can lay the ruler near the line m so that the point D matches with the number 0 and so that the point E matches with a positive number.

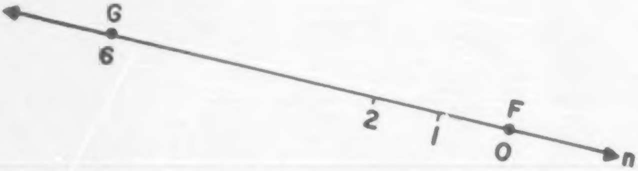


In the example pictured, E matches with 4.5, and the distance DE centimeters is 4.5 cm.

Consider another example in which the same unit-pair is chosen and in which two distinct points F and G belong to the line n in the real world.



A physical ruler, graduated in centimeters, can assign numbers to points on n so that F matches with 0 and G matches with a positive number.



These examples illustrate that, in the real world, no matter what unit-pair is chosen for measuring distance, we may select, on any line, any point as an origin for a number-line and insist that any other given point be assigned a positive number on the number-line. The next postulate is a statement of the corresponding situation in our formal geometry.

Postulate 12. (The Ruler Postulate) If $\{A,A'\}$ is any unit-pair, if ℓ is any line, and if P and Q are any two distinct points on ℓ, then there is a unique coordinate system on ℓ relative to $\{A,A'\}$ such that the origin of the coordinate system is P and the coordinate of Q is positive.

The word "unique" in the statement of Postulate 12 conforms to our experience that there is only one way to lay a physical ruler along a line so that all the conditions are satisfied.

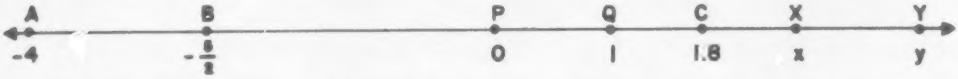
As a by-product, the Ruler Postulate gives us further information about the number of points on a line. Since there are infinitely many real numbers, every line has infinitely many points. This property in formal geometry fits with our idea about a line in the real world. Postulate 2 assured us that each line has two or more points, and until we adopted Postulate 12, we have not been able to be any more specific.

Customarily, when we establish a number-line, we choose a point on the line and assign the number 0 to it, then select another point on the line and assign to it the number 1, and finally assign numbers to other points so that distances are measured relative to the initially chosen two points. Thus, the points to which the numbers 0 and 1 were assigned become a unit-pair. The same type of procedure is permissible in our formal geometry, as the next theorem tells us. We obtain this theorem from the Ruler Postulate by choosing a particular unit-pair, namely, $\{P,Q\}$.

THEOREM 3-1. (The Origin and Unit-Point Theorem) If P and Q are any two distinct points, then there is a coordinate system on the line $\overleftrightarrow{PQ}$ relative to the unit-pair $\{P,Q\}$ such that P is the origin and Q is the unit-point of the coordinate system.

Problem Set 3-5

1. The following diagram is intended to suggest a coordinate system on ℓ relative to $\{P,Q\}$.



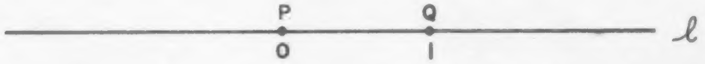
   (a) Which point is the origin? Which point is the unit-point?

   (b) Compute each of the following distances: PQ, BC, AX, CY, XY, CB, YX .

2. Draw a "line" and mark on it two "points" P and Q. Then, on or off the line, mark two points, A and A'. On the line indicate a coordinate system relative to {A,A'} having P as its origin. Record the coordinate of Q in this system. Indicate a second coordinate system, different from the first, but also having P as its origin and {A,A'} as its unit-pair. Compare the coordinates of Q in the two systems. Compute in both coordinate systems the distance PQ. Do the results of your computation agree?

3. Consider two distinct points A, B on line ℓ
   (a) Is there a coordinate system on ℓ whose origin is A and whose unit-point is B ? Is it unique?
   (b) Is there a coordinate system on ℓ whose origin is B and which assigns a positive number to A ? Is it unique?

*4. Suppose that on a line ℓ, a coordinate system assigns the coordinates 0 and 1 to points P and Q, respectively.

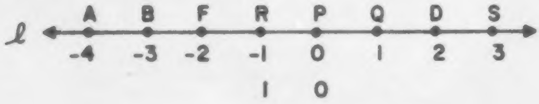


   (a) Consider the set of points of ℓ whose coordinates satisfy the inequality $x > 0$. Likewise, consider the set of all points of ℓ whose coordinates are negative. Do these two sets have any point in common?
   (b) Consider the set of all points of ℓ whose coordinates are greater than 1. Likewise, consider the set of all points of ℓ whose coordinates are less than 1. Do these two sets have any point in common?
   (c) Explain the statement: P plays the same role with respect to the two sets in part (a) that Q plays with respect to the two sets in point (b).

5. In a given coordinate system, find the distances between pairs of points which have the following coordinates:
   (a) $-16, \frac{3}{2}$
   (b) x and y, if $x > y$
   (c) x and y, if $x < y$
   (d) x and y, if $x = y$
   (e) a and b

Problem 5(e) is involved because we do not know which number is larger or even whether they are distinct. Whenever you are asked to find the distance between two points whose coordinates are not known numerically, you should consider the three cases, as in parts (b), (c), (d) in Problem 5. A special symbol, discussed in Problem 10, is often useful in this situation.

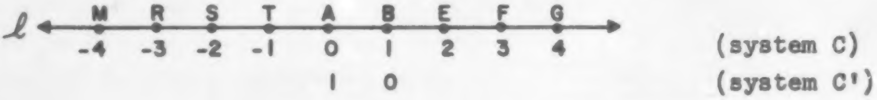
*6. Consider the two coordinate systems, which we may denote by C and C', on line ℓ as suggested in the diagram.



(coordinate system C)
(coordinate system C')

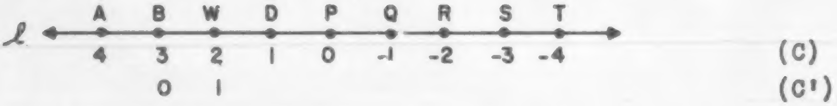
   (a) Copy the diagram and complete the assignment of numbers to the indicated points of ℓ by the coordinate system C'.
   (b) Compare the two correspondences with respect to (1) position of origin; (2) unit-pair used, and (if different) the measure of each relative to the other; (3) whether the unit-point is to the left of the origin or to the right of the origin.
   (c) Note that it appears that the sum of the coordinates assigned to any one point by the two coordinate systems is constant. Using this information and letting x be the coordinate of a point in system C and x' be its coordinate in system C', express a relation between x' and x.
   (d) Compute BR in the C system and BR in the C' system. Compare these numbers.

*7. Two coordinate systems C and C' on line ℓ are suggested in the diagram.



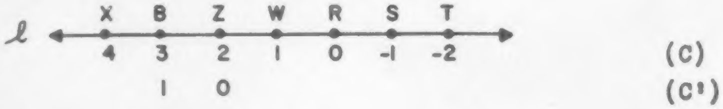
Follow the directions given in parts (a), (b), (c) and (d) in Problem 6.

8. Two coordinate systems C and C' on line ℓ are suggested by the diagram:



Follow the directions given in parts (a), (b), (c) (d) in Problem 6.

9. Two coordinate systems C and C' on line ℓ associate numbers with points as suggested by the diagram:



Follow the directions given in parts (a), (b), (c), (d) in Problem 6. However, in part (c), replace the word "sum" with the word "difference."

10. In order to overcome the difficulty mentioned in Problem 5(e) we find it convenient to introduce the symbol $|a - b|$ which is called the absolute value of the number $a - b$. The symbol, $|a - b|$, names a positive number or zero and this is helpful since every distance is a positive number or zero.

The distance between two points whose coordinates are a and b is $a - b$ if this is not negative; otherwise the distance is $b - a$. Notice that $b - a$ is the same as $-(a - b)$.

The absolute value of a number x is defined by:

$$|x| = x \text{ if } x \geq 0, \text{ and } |x| = -x \text{ if } x < 0.$$

Thus, $|2 - 7| = |7 - 2|$ and $|x - y| = |y - x|$.

It does not matter, in computing distances, whether you subtract the first coordinate from the second or the second from the first as long as you take the distance to be the absolute value of the difference. For example, given two coordinates -6, -2,

$$(-6) - (-2) = -6 + 2 = -4 \quad \text{and} \quad |-4| = 4,$$
$$\text{while} \quad (-2) - (-6) = -2 + 6 = 4 \quad \text{and} \quad |4| = 4.$$

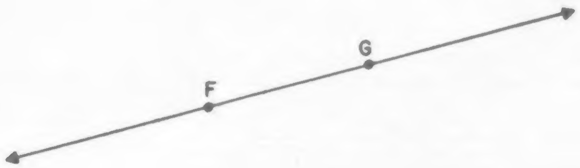
Each method of computation tells us that the distance is 4.

(a) Simplify each of the following:
   (1) $|5|$
   (2) $|-5|$
   (3) $|5 - 4|$
   (4) $|4 - 5|$
   (5) $|(4 - 5) - (5 - 4)|$

(b) In each part of this problem compute the distance between the two points having the given coordinates. Express the answers using the absolute value symbol.
   (1) x and y
   (2) $a + b$ and b
   (3) $a + b$ and $a - b$
   (4) $a + b$ and $b + a$
   (5) $a - b$ and $b - a$

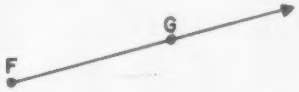
3-6. Rays and Segments.

Two distinct points, F and G, determine (by Postulate 3) a line.

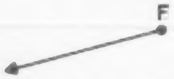


By Postulate 12 there are many other points on $\overleftrightarrow{FG}$. In this section we consider certain subsets of $\overleftrightarrow{FG}$ which are determined by the given points.

Let us also denote $\overleftrightarrow{FG}$ by m. The point F on m appears to separate the line m into two portions. The portion shown in the next diagram contains the point G,

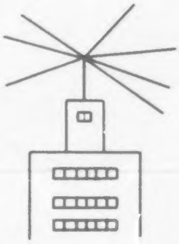


while the other portion, pictured below, does not contain G.



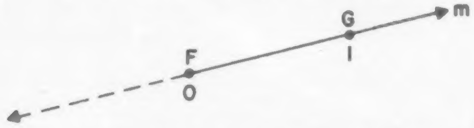
Each of these subsets of m may be considered to have F as an "endpoint."

A subset of a line such as the above may remind us of a physical phenomenon like a ray of light emanating from a source and continuing indefinitely.



We adopt the same word "ray" in geometry to name a set such as we have considered. A coordinate system will help us describe precisely what we mean by a ray.

On the above line m, there is, by Theorem 3-1, a coordinate system such that F is the origin and G is the unit-point.



One of the portions into which m appears to be separated by F consists of F and all the points with positive coordinates.

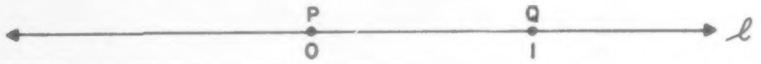
DEFINITION. If A and B are distinct points, then, in the coordinate system on $\overleftrightarrow{AB}$ with origin A and unit-point B, the subset of $\overleftrightarrow{AB}$ consisting of all points whose coordinates x satisfy $x \geq 0$ is called a ray, or more specifically, the ray with endpoint A and containing B.

Notation. If A and B are distinct points, the ray with endpoint A and containing B is denoted by the symbol $\overrightarrow{AB}$.

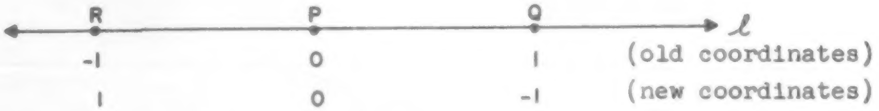
We emphasize that in the symbol $\overrightarrow{AB}$ the first-named point is the endpoint of the ray.

THEOREM 3-2. Every point on a given line is the endpoint of two rays on the line, and the intersection of these two rays is the point itself.

Proof: Let P be a point on the line ℓ. If Q is another point on ℓ, there is, by Theorem 3-1 (the Origin and Unit-Point Theorem) a coordinate system on ℓ in which P is the origin and Q is the unit-point.



The ray $\overrightarrow{PQ}$ has endpoint P and consists of all points whose coordinates satisfy $x \geq 0$. If R is the point with coordinate -1, then R does not belong to $\overrightarrow{PQ}$. By Theorem 3-1, there is another coordinate system on ℓ with origin P and with unit-point R.



Using the new coordinate system, we find that $\overrightarrow{PR}$ is a ray given by the inequality $x' \geq 0$. Since, by Postulate 11, the new coordinate x' and the old coordinate x of a point are related by the equation

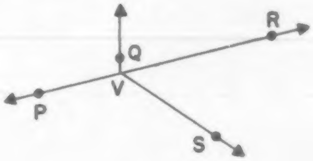
$$x' = -x,$$

the ray $\overrightarrow{PR}$ is the set of all points whose original coordinates x satisfy $x \leq 0$. The intersection of the distinct rays $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ consists of P alone.

It is convenient to have a name for two rays which are related in the manner described in the preceding theorem.

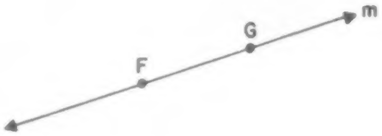
> DEFINITION. Two distinct collinear rays with a common endpoint are called opposite rays, and each ray is said to be opposite to the other.

In our future work we shall often consider two rays which have the same endpoint, but which are not opposite rays because they are not collinear. Or we may consider more than two rays, all of which have the same endpoint. The diagram shows four rays, each with endpoint V.

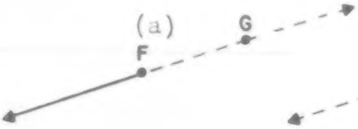


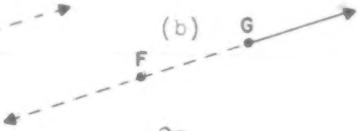
> DEFINITION. Rays all of which have the same endpoint are called concurrent rays.

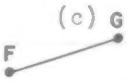
Now let us consider another type of subset determined by two distinct points. Suppose that the points F and G determine the line m.



The diagram suggests that the pair of points split the line into three parts which do not overlap: the ray with endpoint F which does not contain G (drawn heavily in Figure (a)), the ray with endpoint G which does not contain F (drawn heavily in Figure (b)), and the subset pictured in Figure (c).







We may conveniently use coordinates to describe a set of the type shown in Figure (c).

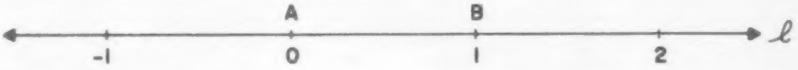
DEFINITIONS. If A and B are distinct points, then, in the coordinate system on $\overleftrightarrow{AB}$ with origin A and unit-point B, the subset of $\overleftrightarrow{AB}$ consisting of all points whose coordinates x satisfy $0 \leq x \leq 1$ is called the segment joining the given points.

Each of the points joined by a segment is called an endpoint of the segment.

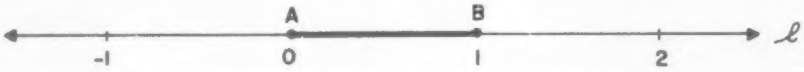
Notation. If A and B are distinct points, the segment joining these two points is denoted by the symbol $\overline{AB}$.

Since each of the symbols $\overline{AB}$ and $\overline{BA}$ names the segment joining the two points, $\overline{AB}$ and $\overline{BA}$ are different names for the same segment. Thus we conclude that $\overline{AB} = \overline{BA}$ for any two distinct points A, B.

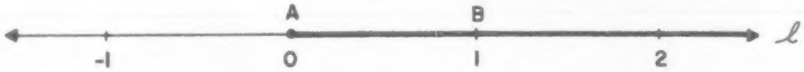
Example 1. Consider the coordinate system on the line ℓ shown in the diagram.



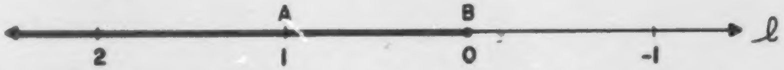
(a) The segment $\overline{AB}$ is the set of points whose coordinates x satisfy $0 \leq x \leq 1$ and is marked heavily in the diagram below.



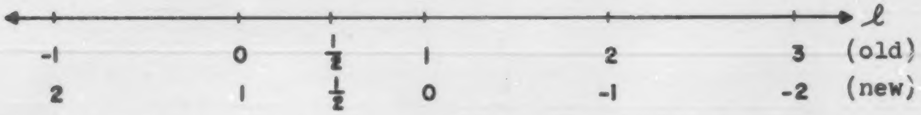
(b) The ray $\overrightarrow{AB}$ is the set of points whose coordinates x satisfy $x \geq 0$ and is marked heavily in the next diagram.



(c) We find the ray $\overrightarrow{BA}$ by choosing a coordinate system on ℓ with origin B and unit-point A. If x' is the new coordinate of a point, then $\overrightarrow{BA}$ is the set of points whose (new) coordinates x' satisfy $x' \geq 0$.



(d) What is the relationship between the old coordinate x and the new coordinate x' of a point on ℓ ? We mark the same diagram with both old and new coordinates, and compare them.



Note that the sum $x + x'$ is, for every point, the same, namely 1. Thus

$$x + x' = 1,$$

or

$$x' = 1 - x.$$

(e) Since $\overrightarrow{BA}$ is associated with the inequality $x' \geq 0$ and since $x' = 1 - x$, what inequality in the old coordinate system is associated with $\overrightarrow{BA}$? Replacing x' by $1 - x$, we obtain

$$1 - x \geq 0 .$$

By the additive property of order,

$$1 \geq x, \quad \text{or} \quad x \leq 1.$$

(f) Since $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are associated respectively with the inequalities $0 \leq x$ and $x \leq 1$, their intersection is associated with $0 \leq x \leq 1$.

(g) The intersection of the two rays $\overrightarrow{AB}$ and $\overrightarrow{BA}$ is the segment $\overline{AB}$.

As a summary, it may be helpful to review, on a line, six important subsets determined by two points of the line. If we call the points A and B, the following six pictures show the same line, with the subset marked heavily.

Segment $\overline{AB} = \overline{BA}$

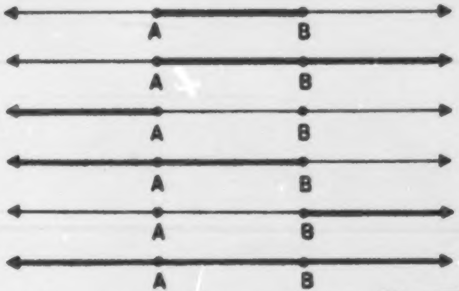
Ray $\overrightarrow{AB}$

Ray opposite to $\overrightarrow{AB}$

Ray $\overrightarrow{BA}$

Ray opposite to $\overrightarrow{BA}$

Line $\overleftrightarrow{AB} = \overleftrightarrow{BA}$



Notice carefully the distinction among the symbols $\overline{AB}$, $\overrightarrow{AB}$, $\overleftrightarrow{AB}$, $\overrightarrow{BA}$.

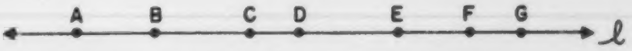
Problem Set 3-6

1. Name the origin and unit-point which may be used in defining each of the rays and segments below.

(a)	$\overrightarrow{CF}$	(e)	$\overrightarrow{BE}$
(b)	$\overline{BG}$	(f)	$\overrightarrow{CA}$
(c)	$\overrightarrow{ED}$	(g)	$\overline{BE}$
(d)	$\overline{CF}$	(h)	$\overline{DE}$

Is there any choice possible in the origin and unit-point?

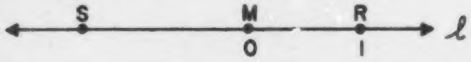
2. Points on line ℓ are indicated in the diagram.



Using this diagram, name the following:

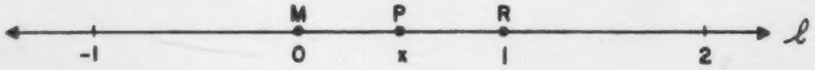
(a) Intersection of $\overrightarrow{CF}$ and $\overline{CF}$
(b) Union of $\overrightarrow{CF}$ and $\overline{CF}$
(c) Intersection of $\overrightarrow{CF}$ and $\overrightarrow{CA}$
(d) Union of $\overrightarrow{CF}$ and $\overrightarrow{CA}$
(e) Intersection of $\overrightarrow{BE}$ and $\overrightarrow{ED}$
(f) Union of $\overline{BE}$ and $\overline{CF}$
(g) Intersection of $\overline{BE}$ and $\overline{CF}$
(h) Intersection of $\overrightarrow{CA}$ and $\overrightarrow{DE}$

3. Name, in terms of the given letters on the indicated line ℓ, the segment or ray which is the set of all points whose coordinates satisfy the given inequality.



   (a) $x \geq 0$
   (b) $x \leq 0$
   (c) $0 \leq x \leq 1$

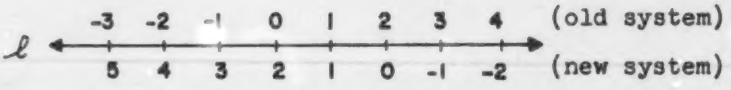
4. Let M and R be any distinct points on a line ℓ :
   (a) Is there a coordinate system on ℓ in which 0 is assigned to M and 1 to R ?



   Consider in $\overline{MR}$ a point P distinct from M and R and with coordinate x.

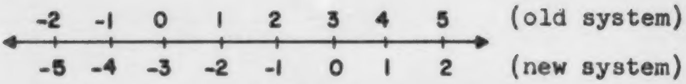
   (b) Between what numbers must x lie?
   (c) Write an equation involving x which states that PM = PR.
   (d) What is the solution set of this equation? How many numbers are in this solution set?
   (e) How many points P are in $\overline{MR}$ such that MP = PR ?

*5. Two coordinate systems on line ℓ are indicated in the diagram. One is called the old, the other the new coordinate system.



From the fact that the sum of the old and new coordinates appears to be the same for all points, find an expression for the new coordinate x' of a point on ℓ in terms of its old coordinate x. Does the diagram remind you of two rulers placed against the same line?

*6.



Examine the diagram in a manner similar to that used in Problem 5, but note that this time it is the *difference* of the coordinates which should be considered. Find an expression for the new coordinate x' in terms of the old coordinate x. How does the placement of the two rulers, as suggested here, differ from that indicated in Problem 5?

3-7. *Interior Points*.

A ray has one endpoint and a segment has two endpoints. Sometimes we are primarily interested in those points of a ray or segment which are not endpoints.

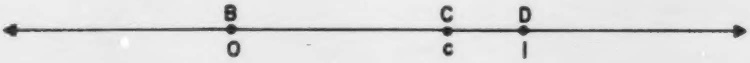
DEFINITIONS. The *interior of a ray* is the set of all points of the ray except the endpoint.

Each point in the interior of a ray is called an *interior point* of the ray.

The *interior of a segment* is the set of all points of the segment except the two endpoints.

Each point in the interior of a segment is called an *interior point* of the segment.

Since the interior of a ray or segment is distinguished from the ray or segment only by the omission of endpoints, we may express interiors in terms of a coordinate system.



The interior of $\overrightarrow{BD}$ is the set of points whose coordinates satisfy $x > 0$. The interior of $\overline{BD}$ is the set of points whose coordinates satisfy $0 < x < 1$.

Consider a point C in the interior of $\overline{BD}$. In the above coordinate system on $\overline{BD}$, the coordinate c of the point C satisfies $0 < c < 1$. The coordinate of C is between the coordinate 0 of B and the coordinate 1 of D. It is convenient to shorten the preceding sentence and say simply that C is between B and D.

> *DEFINITION*. A point is said to be *between* the distinct points B and D if and only if it is an interior point of $\overline{BD}$.

Notice that the statement that one point is between two points guarantees that the three points are distinct and are collinear.

In many situations the following convention will be useful. If we speak of three distinct points P, Q, R as being "collinear", we are not saying which one of the three is between the other two. But if we speak of points P, Q, R as being *collinear in that order*, we shall mean that the points are distinct and that Q is between P and R. Similarly, if we say that four points P, Q, R, S are *collinear in that order*, we shall mean both that Q is between P and R and that R is between Q and S.

Among the interior points of a segment, the so-called midpoint has special importance.

> *DEFINITION*. The *midpoint* of a segment is the point which belongs to the segment and is equally distant from the endpoints of the segment.

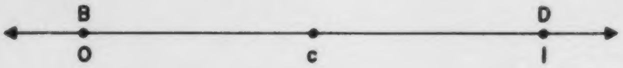
In symbols, if E and F are distinct points, then the midpoint M of $\overline{EF}$ is the point which fulfills two requirements,

namely, that M is between E and F and that $EM = MF$.

Before using the notion of a midpoint as defined, we should first assure ourselves that every segment has a midpoint and moreover only one midpoint. The next theorem gives us this information.

THEOREM 3-3. Every segment has a unique midpoint.

Proof: Let $\overline{BD}$ be any segment. Consider the coordinate system on $\overleftrightarrow{BD}$ with origin B and unit-point D. Then $\overline{BD}$ is the set of all points whose coordinates x satisfy $0 \leq x \leq 1$. Let c be the coordinate of the desired midpoint. Then $0 \leq c \leq 1$, because the midpoint belongs to the segment. Using the coordinate system to compute distances,



we find that $c - 0 = 1 - c$, since the midpoint is equally distant from the endpoints. The equation $c = 1 - c$ has exactly one solution, namely, $c = \frac{1}{2}$. Thus the one and only one midpoint of $\overline{BD}$ is the unique point whose coordinate is $\frac{1}{2}$.

DEFINITION. The midpoint of a segment is said to *bisect* the segment. More generally, any set of points whose intersection with a segment consists only of the mid-point of the segment is said to *bisect* the segment.

The point which bisects the segment $\overline{EF}$ is the point M between E and F such that $EM = \frac{1}{2} \cdot EF = MF$.

Problem Set 3-7

1. Given a coordinate system on $\overleftrightarrow{AB}$ such that A is the origin and B is the unit-point,
   - (a) Give the coordinate of the endpoint of $\overrightarrow{AB}$.
   - (b) The interior of $\overline{AB}$ is the set of all points whose coordinates x satisfy what inequality?
   - (c) The ray opposite to $\overrightarrow{AB}$ is the set of points whose coordinates x satisfy what inequality?
   - (d) Find the coordinates of points C, D, E on $\overrightarrow{AB}$, such that $BC = 4$; $AD = 4$; and $AE = EB$.
   - (e) If P is between A and B, what values can its coordinate have?
   - (f) Let x be between A and B.
     - (1) May A, B, and X be noncollinear?
     - (2) Is X in $\overline{AB}$?
     - (3) Is X in the interior of $\overline{AB}$?
     - (4) Can the coordinate of X be negative?
     - (5) Can $BA = AX$?
     - (6) Can B be between A and X ?
   - (g) Fill the blanks:

     The set of points whose coordinates satisfy $x < 0$ is the _____ of the _____ which is _____ to $\overrightarrow{AB}$.

*2. Fill in the blanks in the following table. (The names "alpha" and "beta" are used here as arbitrary names for units.)

PQ (in inches) = 3	RS (in inches) =
PQ (in feet) =	RS (in feet) = $\frac{1}{2}$
PQ (in yards) =	RS (in yards) =
PQ (in alphas) = 2	RS (in alphas) =
PQ (in betas) = 8	RS (in betas) =
TV (in inches) =	VW (in inches) = 24
TV (in feet) =	VW (in feet) =
TV (in yards) = $\frac{1}{2}$	VW (in yards) =
TV (in alphas) =	VW (in alphas) =
TV (in betas) =	VW (in betas) =

*3. Fill in each of the blanks with a number. You may wish to use the data from Problem 2 to help you answer. (Note that the dot after each blank is a symbol for multiplication.)

(a) PQ (in feet) = ___?___ · PQ (in yards).
(b) RS (in feet) = ___?___ · RS (in yards).
(c) TV (in inches) = ___?___ · TV (in yards).
(d) TV (in yards) = ___?___ · TV (in inches).
(e) RS (in inches) = ___?___ · RS (in feet).
(f) VW (in feet) = ___?___ · VW (in inches).
(g) RS (in yards) = ___?___ · RS (in feet).

4. (a) In measuring any distance, say MR, the longer the unit used the ________ the measure.

(b) From Problem 2, the measure of PQ (in betas) is _______ times the measure of PQ (in alphas).

(c) If the distance between two points measured in alphas is 1, and if the distance between two other points measured in betas is 1, which pair of points is farther apart? How many times farther apart?

*5. Refer to Problem 2 to fill in the blanks below:

(a) VW (in inches) is 8 times PQ (in inches) ;

$$\frac{\text{VW (in inches)}}{\text{PQ (in inches)}} = 8$$

(b) VW (in feet) is _____ times PQ (in feet) ;

$$\frac{\text{VW (in feet)}}{\text{PQ (in feet)}} = \underline{\qquad}$$

(c) VW (in yards) is _____ times PQ (in yards) ;

$$\frac{\text{VW (in yards)}}{\text{PQ (in yards)}} = \underline{\qquad}$$

(d) If AB is 10 times CD when an inch is used as a unit, then AB will be how many times CD when any other unit is used?

*6. Given points D, F, I, N, T and Y such that
FT (in feet) = 1, IN (in inches) = 1, YD (in yards) = 1.
Find the measure of each of the following:

(a) FT (in inches)
(b) IN (in feet)
(c) YD (in feet)
(d) FT (in yards)
(e) YD (in inches)
(f) IN (in yards)

*7. Given two distinct points P and Q, which of the measures in Problem 6 is equivalent to

(a) $\frac{PQ \text{ (in inches)}}{PQ \text{ (in feet)}}$

(b) $\frac{PQ \text{ (in feet)}}{PQ \text{ (in inches)}}$

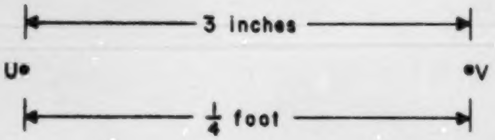
(c) $\frac{PQ \text{ (in inches)}}{PQ \text{ (in yards)}}$

3-8. Relationship Between Distances Relative to Different Unit-Pairs.

On a given line there are many different coordinate systems. We have taken advantage of this fact throughout the preceding sections. Often we have chosen a coordinate system to suit our purposes. In a few situations we have used more than one coordinate system in a single discussion. Let us review why we have been justified in making these choices. According to the Ruler Postulate, we are at liberty to select any point whatsoever on the given line as an origin. We also are permitted to choose freely the unit-pair for measuring distances. How do the different coordinate systems on the same line compare with one another? Sections 3-8 and 3-9 are devoted to answering this question.

As a first step, it is desirable to investigate more fully how the idea of distance depends upon the choice of a unit-pair. In the real world, we are familiar with many different standards of measurement: the inch, the foot, the mile, the meter, and many others. Although it is possible to use only one unit, it does not seem practical to do so. It is no more sensible to express the distance to the moon in inches than it is to measure the length of a needle in miles. What, then, we may ask, is the effect on our definitions and theorems if we choose to use a different unit-pair?

Refer to Problems 6 and 7 in Problem Set 3-7. As shown in the diagram, if two points U and V are 3 inches apart, then they are $\frac{1}{4}$ foot apart.

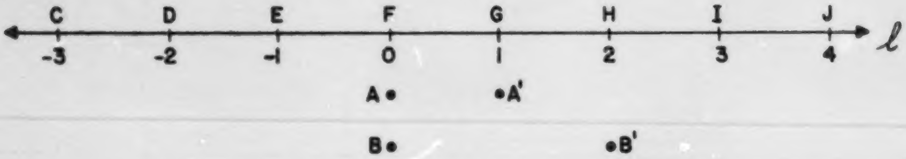


Similarly, if AB yards = 5 yds., then AB feet = 15 ft. Thus, $\frac{\text{AB (in feet)}}{\text{AB (in yards)}} = \frac{15}{5} = 3$. If P and Q are any two distinct points, then $\frac{\text{PQ (in feet)}}{\text{PQ (in yards)}}$ is the number 3. We notice that 3 is the distance, in feet, between points which are one yard apart; that is, 3 is the distance, in feet, between a pair of points which determine the yard.

Likewise, if R and S are any two distinct points, then $\frac{\text{RS (in meters)}}{\text{RS (in centimeters)}}$ is a constant. If RS cm. = 350 cm., then RS meters = 3.5 m., and the constant is $\frac{3.5}{350} = \frac{1}{100}$. The number $\frac{1}{100}$ is the distance, in meters, between two points which serve as a unit-pair for the centimeter. These examples from the physical world suggest the following postulate, which explains in our formal geometry how distances, relative to different unit-pairs, compare with one another.

> Postulate 13. Let A and A' be any two distinct points and let B and B' be any two distinct points. Then, for every pair of distinct points P and Q in space, $\frac{\text{PQ(relative to } \{A,A'\})}{\text{PQ(relative to } \{B,B'\})}$ is a constant.

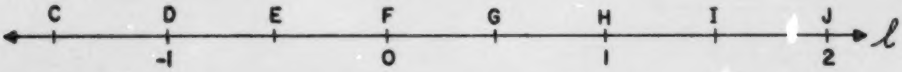
Example 1. On a line ℓ suppose that a coordinate system relative to the unit-pair (A,A') is given:



Suppose that (B,B') is another unit-pair and that

$$BB' \text{ (relative to } (A,A')) = 2.$$

On the same line ℓ, we also have a coordinate system relative to (B,B') and with the same point F as origin, as shown in the next diagram.



Computing with coordinates in the two systems, we obtain

$$\frac{DJ \text{ (relative to } (A,A'))}{DJ \text{ (relative to } (B,B'))} = \frac{4-(-2)}{2-(-1)} = \frac{6}{3} = 2,$$

a number which, we note, is the same as BB' (relative to (A,A')).

Postulate 13 states that $\dfrac{PQ \text{ (relative to } (A,A'))}{PQ \text{ (relative to } (B,B'))}$ is a constant, say k, not depending on P nor Q. In particular, then, if we choose B and B' as P and Q, we find that

$$k = \frac{BB' \text{ (relative to } (A,A'))}{BB' \text{ (relative to } (B,B'))}$$

$$= \frac{BB' \text{ (relative to } (A,A'))}{1}$$

$$= BB' \text{ (relative to } (A,A')).$$

As in the examples discussed above, the constant k is the measure of the distance between the points of one unit-pair, relative to the other unit-pair.

Example 2. Let A, B, C, D, E, F be six distinct points, let r be a positive number, and consider two unit-pairs for measuring distances. Fill the blanks in the following table.

	relative to: first unit-pair	 second unit-pair
Distance AB	5	20
Distance CD	? ? ?	8
Distance EF	r	? ? ?

Postulate 13 states that $\frac{PQ\text{ (first unit-pair)}}{PQ\text{ (second unit-pair)}}$ is a constant, say k. Using A and B as the two points, we find $\frac{5}{20} = k$.

(a) If we let $x = CD$ (first unit-pair), then $\frac{x}{8} = k = \frac{1}{4}$. Hence, $x = 2$.

(b) If we let $y = EF$ (second unit-pair), then $\frac{r}{y} = k = \frac{1}{4}$. Hence, $y = 4r$.

Example 3. In Example 2(a), we found that $\frac{5}{20} = k = \frac{2}{8}$. We may also say that $\frac{5}{2} = \frac{20}{8}$. Noticing the positions of the numbers 5, 20, 2, 8 in the table, we conclude that:

$$\frac{AB\text{ (first unit-pair)}}{CD\text{ (first unit-pair)}} = \frac{AB\text{ (second unit-pair)}}{CD\text{ (second unit-pair)}}.$$

Example 3, together with Problem 5 in Problem Set 3-6(b), suggests the following theorem. The theorem states that the distance between one pair of distinct points divided by the distance between another pair of distinct points is the same, regardless of the choice of unit-pair for measuring distances.

THEOREM 3-4. Let $\{A,A'\}$ and $\{B,B'\}$ be any unit-pairs, let M and N be any two distinct points, and let E and F be any two distinct points. Then

$$\frac{MN\text{ (relative to }\{A,A'\})}{EF\text{ (relative to }\{A,A'\})} = \frac{MN\text{ (relative to }\{B,B'\})}{EF\text{ (relative to }\{B,B'\})}.$$

Proof: If we let

r = MN (relative to {A,A'}); s = MN (relative to {B,B'});
u = EF (relative to {A,A'}); v = EF (relative to {B,B'});

then we must show that

$$\frac{r}{u} = \frac{s}{v} .$$

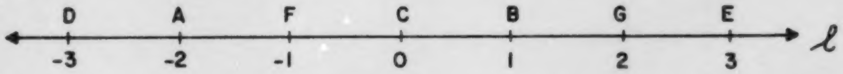
By Postulate 13, $\frac{r}{s}$ is the same number as $\frac{u}{v}$.

Since $$\frac{r}{s} = \frac{u}{v} ,$$

we find that $$\frac{r}{u} = \frac{s}{v} .$$

Problem Set 3-8

1. Given the coordinate system on ℓ indicated by the drawing.



(a) Find the following distances:

FG (relative to {C,B})
FA (relative to {C,G})
DB (relative to {D,B})
AG (relative to {B,E})

(b) If X is a point to the right of C, what letter should X be in each of the following to product a correct statement?

GE (relative to {C,X}) = $\frac{1}{2}$

DF (relative to {C,X}) = 2

FG (relative to {C,X}) = 1

FG (relative to {C,X}) = $\frac{3}{2}$

(c) Show that

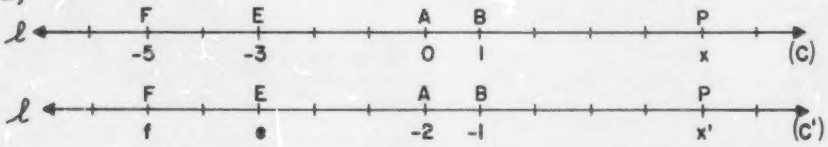
$$\frac{\text{FB (relative to \{A,G\})}}{\text{FB (relative to \{C,B\})}} = \frac{\text{BG (relative to \{A,G\})}}{\text{BG (relative to \{C,B\})}}$$

$$= \text{CB (relative to \{A,G\})}$$

(d) Verify that

$$\frac{DF\ (\text{relative to } \{C,B\})}{GE\ (\text{relative to } \{C,B\})} = \frac{DF\ (\text{relative to } \{F,G\})}{GE\ (\text{relative to } \{F,G\})}$$

*2. (a)



The diagram shows two different pictures of one line ℓ. Five points on ℓ are named. Two coordinate systems C and C' on ℓ are indicated in the different pictures. Our task is to find the coordinate in C' for each point for which the coordinate in C is given.

(1) Compute $\frac{AB\ (\text{in } C')}{AB\ (\text{in } C\ )}$.

(2) Using the information in (1), what number is $\frac{BE\ (\text{in } C')}{BE\ (\text{in } C\ )}$?

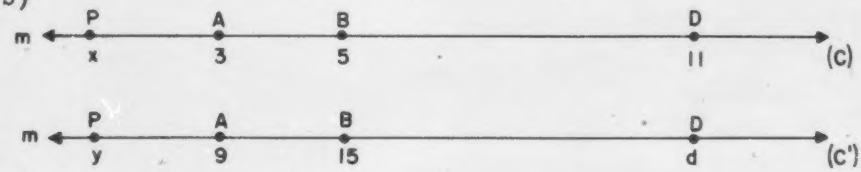
(3) Using the coordinates of B and E in the two systems, compute $\frac{BE\ (\text{in } C')}{BE\ (\text{in } C\ )}$.

(4) Combine the results of (2) and (3) to obtain an equation for e, and solve for e.

(5) $\frac{AP\ (\text{in } C')}{AP\ (\text{in } C\ )}$ = __________. Write an equation in terms of coordinates, and solve the equation for x' in terms of x.

(6) By using the results of (5), find the number in the system C' which corresponds to -5 in the system C.

(b)



Two coordinate systems C and C' on line m assign numbers to points as indicated in the two pictures of the line m.

(1) Compute $\frac{AB \text{ (in } C')}{AB \text{ (in } C)}$.

(2) Using the information in (1), what number is $\frac{AD \text{ (in } C')}{AD \text{ (in } C)}$?

(3) Using the result of (2), and the two coordinate systems, obtain an equation for d, and solve for d.

(4) $\frac{AP \text{ (in } C')}{AP \text{ (in } C)} =$ __________. Write an equation in terms of coordinates, and solve for y in terms of x.

(5) Using the result of (4), find the coordinate in C' of P if its coordinate in C is -8.

*3. Given a coordinate system C on a line ℓ. Let x be the coordinate of point X in the system. Let another correspondence C' between ℓ and the set of real numbers match x' with point X such that for each x, $x' = 3x$.

(a) Is C' a one-to-one correspondence?

(b) Is C' a coordinate system; that is, does it satisfy the definition?

(c) Is the origin the same point in both systems?

(d) Is the unit-point the same in both cases?

*4. Answer the same questions as in Problem 3, if for each x, $x = x + 5$.

5. Suppose that P, Q, R, S, T, V are points such that, relative to the unit-pair $\{A,A'\}$, $PQ + RS = TV$. Is it true that $PQ + RS = TV$ relative to any other unit-pair $\{B,B'\}$? Explain.

3-9. Relationship Between Different Coordinate Systems on a Line.

Now we are ready to explain the relationship between two coordinate systems on a line. We consider an example.

Let ℓ be a line. Consider two coordinate systems, which we may call C and C', on the line. Suppose R, S, T, X are points on ℓ with coordinates $4, 6, 13, x$, respectively, in system C, and with coordinates $-3, -9, t', x'$, respectively, in C'. Find t' and x'.

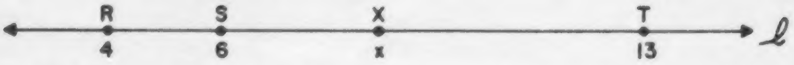


Diagram of line with coordinate system C

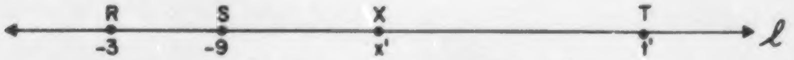


Diagram of line with coordinate system C'

Using the Ruler Postulate we find that RS (in C) $= 6 - 4 = 2$ and RS (in C') $= (-3) - (-9) = 6$. Since $\frac{6}{2} = 3$, we conclude from Postulate 13 that the distance in C' between any two points on ℓ is 3 times the distance in C between the same two points.

We proceed to find t'. In the left column we try to find t' by using the points R and T, in the right column by using the points S and T. In both columns we use the property that a distance in C' is 3 times the corresponding distance in C, and we make repeated use of the Ruler Postulate.

RT (in C) $= 13 - 4 = 9$	ST (in C) $= 13 - 6 = 7$
RT (in C') $= 3 \cdot 9 = 27$	ST (in C') $= 3 \cdot 7 = 21$
RT (in C') $= t' - (-3)$ if $t' > -3$,	ST (in C') $= t' - (-9)$ if $t' > -9$,
or $(-3) - t'$ if $-3 > t'$	or $(-9) - t'$ if $-9 > t'$
Therefore,	Therefore,
$27 = t' - (-3)$ or $27 = (-3) - t'$	$21 = t' - (-9)$ or $21 = (-9) - t'$
$t' = 24$ or $t' = -30$	$t' = 12$ or $t' = -30$

Since the conclusions in both of these columns must be true, it follows that $t' = -30$.

We now use a similar procedure to find x' in terms of x.

RX (in C) $= x - 4$ or $4 - x$
RX (in C') $= 3(x - 4)$ or $3(4 - x)$

On the other hand,

RX (in C') $= x' - (-3)$ or $(-3) - x'$

Therefore,

(1) $3(x - 4) = x' - (-3)$, or
(2) $3(x - 4) = (-3) - x'$, or
(3) $3(4 - x) = x' - (-3)$, or
(4) $3(4 - x) = (-3) - x'$.

Solving each of these four equations for x', we find that there are just two possibilities:

(5) $x' = 3x - 15$,
from (1) and (4); or
(6) $x' = 3x + 9$,
from (2) and (3).

SX (in C) $= x - 6$ or $6 - x$
SX (in C') $= 3(x - 6)$ or $3(6 - x)$

On the other hand,

SX (in C') $= x' - (-9)$ or $(-9) - x'$

Therefore,

(7) $3(x - 6) = x' - (-9)$, or
(8) $3(x - 6) = (-9) - x'$, or
(9) $3(6 - x) = x' - (-9)$, or
(10) $3(6 - x) = (-9) - x'$.

Solving each of these four equations for x', we find that there are just two possibilities:

(11) $x' = 3x - 9$,
from (7) and (10); or
(12) $x' = -3x + 9$,
from (8) and (9).

Note that equation (6) is the same as equation (12). It is impossible for both equation (5) and equation (11) to be true. (If they were, then $3x - 15 = 3x - 9$ and $15 = 9$ would be true. But we know that $15 = 9$ is false.) Since the conclusion in both columns must be true, it follows that $x' = -3x + 9$.

The above reasoning applies to the coordinates of any point X on ℓ and hence $x' = -3x + 9$ is true for every point on ℓ. As a check, for the point R we have $x = 4$ and $x' = -3$. Substituting into $x' = -3x + 9$, we obtain $(-3) = -3 \cdot 4 + 9$, which is true. We leave it for you to check the point S.

This example suggests the relationship we want. The result is stated in the following theorem whose proof is indicated in our example.

THEOREM 3-5. (The Two Coordinate Systems Theorem) Let a line ℓ and two coordinate systems, C and C', on ℓ be given. There exist two numbers a, b, with $a \neq 0$, such that for any point on ℓ, its coordinate x in C is related to its coordinate x' in C' by the equation $x' = ax + b$.

Problem Set 3-9

1. On line ℓ in coordinate system C, the coordinates of T, R, S are, respectively, -1, 0, 2. In coordinate system C', the coordinates of R, S are 0, 4, respectively. Following the steps outlined below, find the C' coordinate of T.

   (a) Compare the distance RS (in C') to the distance RS (in C).

   (b) What is the distance RT (in C) ?

   (c) On the basis of the answers to (a) and (b), what is the distance RT (in C') ?

   (d) On the basis of the answer to (c), what are the two possibilities for the coordinate of T (in C') ?

   (e) What is the distance ST (in C) ?
   the distance ST (in C') ?

   (f) On the basis of the answer to (e), what are the two possibilities for the coordinate of T (in C') ?

   (g) Can you decide which of these is the coordinate of T in C' without appealing to a picture? How?

2. On line ℓ in coordinate system C, the coordinates of T, R, S are respectively, -1, 0, 2. In coordinate system C', the coordinates of R, S are 0, 4, respectively. Following the steps outlined below, find the C' coordinate of T.

   (a) From the Two Coordinate Systems Theorem we know that the coordinate x' of a point in the C' system is related to the coordinate x of that point in the C system by the equation: $x' = ax + b$.

   Hence, since the coordinate of R is 0 in C and 0 in C', we know that

   $$0 = a \cdot 0 + b.$$

   Similarly, since the coordinate of S is 2 in C and 4 in C', we have

   $$4 = a \cdot 2 + b.$$

   From these two equations, determine the values of a and b.

(b) Using a and b determined above, and the formula $x' = ax + b$, determine the coordinate of T in the C' system.

3. On line ℓ in coordinate system C, the coordinates of T, R, S are, respectively, $3, 2, 8$. In coordinate system C', the coordinates of R, S are $1, -2$, respectively. Following the steps outlined in Problem 1, find the C' coordinate of T.

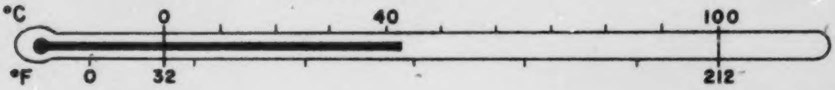
4. On line ℓ in coordinate system C, the coordinates of T, R, S are, respectively, $3, 2, 8$. In coordinate system C', the coordinates of R, S are $1, -2$, respectively. Following the steps outlined in Problem 2, find the C' coordinate of T.

5. On line ℓ in coordinate system C, the coordinates of P, Q, W are $5, 11, -2$, respectively. In coordinate system C', the coordinates of P, Q are $3, 1$, respectively.

   (a) Determine the coordinate of W in C' without using the Two Coordinate Systems Theorem.

   (b) Determine the coordinate of W in C' by using the Two Coordinate Systems Theorem.

6. Assume that the Fahrenheit and Centigrade scales are applied to a thermometer in the manner that two coordinate systems are applied to the same line.



   (a) Find the Fahrenheit reading corresponding to $40^\circ C$.

   (b) Find the Centigrade reading corresponding to $0^\circ F$.

   (c) Use the Two Coordinate Systems Theorem to find the relation between Fahrenheit and Centigrade readings.

7. From the Two Coordinate Systems Theorem we know that two coordinate systems C' and C on a line ℓ are related by $x' = ax + b$. To see more clearly the significance of a and b in this formula, consider the following:

   (a) What is the coordinate in the C' system of the point whose coordinate in the C system is zero?

(b) What is the coordinate in the C' system of the point whose coordinate in the C system is one?

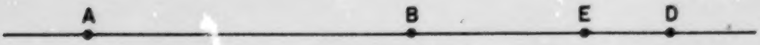
(c) Draw a diagram illustrating the information obtained so far.

(d) Compare the distance between the two points mentioned in (a) and (b) in the C system and the distance between them in the C' system.

(e) Discuss the sign of a as it relates to whether the C' and C coordinates are like two rulers placed against ℓ so that their scales "run in the same direction" or are like two rulers placed against ℓ so that their scales "run in opposite directions." Draw some diagrams to illustrate the possibilities.

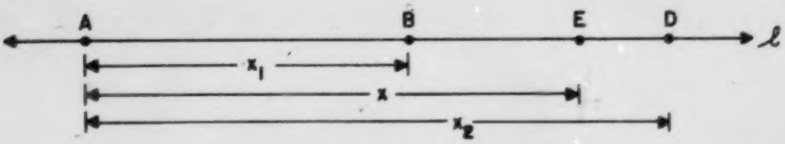
(f) Can you give a reason why a must be different from zero in the Two Coordinate Systems Theorem?

*8. (a) Four towns are located on a straight road as indicated in the drawing.



Town B is 5 miles east of town A, town D is 9 miles east of town A, and town E is $\frac{2}{3}$ of the way from B to D. How far from A is E ? Is it reasonable to think of the distance from A to E as being the distance from A to B plus a fraction of the distance from B to D ?

(b) Four points A, B, E, D are collinear in that order.



The distance between A and B is x_1, the distance between A and D is x_2, and E is $\frac{2}{3}$ of the way from B to D. Find the distance, say x, between A and E in terms of x_1 and x_2. (Hint: Think of the distance AE as the sum of the distance AB and a certain fraction of the distance BD.)

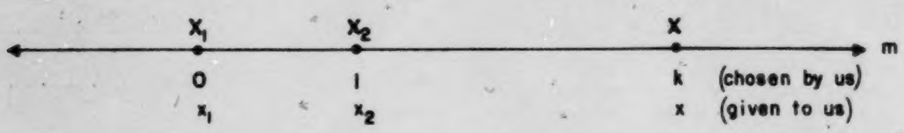
(c) Suppose that a coordinate system C on line ℓ in part (b) assigns coordinates 0 and 1 to B and D, respectively. Explain why the coordinate of E is $\frac{2}{3}$.

(d) Suppose that another coordinate system C' on line ℓ in part (b) assigns coordinates x_1 and x_2 to B and D, respectively. Find the coordinate x of E in the coordinate system C'. (Your answer should be in terms of x_1 and x_2.)

3-10. Using a Given Coordinate System.

Let X_1 and X_2 be two distinct points, and let m be the line which they determine. Suppose that a coordinate system on m is given. In this system suppose the coordinates of X_1 and X_2 are x_1 and x_2, respectively. By the Origin and Unit-Point Theorem, we may choose a coordinate system on m in which X_1 and X_2 have respective coordinates 0 and 1. As an important application of the preceding section we wish to find the relationship between these two coordinate systems on $\overleftrightarrow{X_1X_2}$; that is, between the system given to us and the system chosen by us.

As a convenience in later work, we shall use the letters x and k in place of the symbols x' and x in the statement of Theorem 3-5. That is, for any point X on m, its coordinate in the given system is x and its coordinate in the specially chosen system is k. The diagram shows both coordinate systems.



With this notation, the Two Coordinate Systems Theorem states that x and k are related by the formula

$$x = ak + b,$$

where a and b are certain numbers. We wish to find a and b. At the point X_1,

$$x_1 = a \cdot 0 + b;$$

hence,

$$b = x_1 .$$

At the point X_2,

$$x_2 = a \cdot 1 + b = a + x_1 ;$$

hence,

$$a = x_2 - x_1 .$$

Thus the formula is

$$x = (x_2 - x_1)k + x_1 .$$

This result which we have just proved is so important in later chapters that we will restate it as the next theorem.

THEOREM 3-6. (The Two-Point Theorem) In any coordinate system on a line ℓ, let x_1 and x_2 be the respective coordinates of distinct points X_1 and X_2 on ℓ. Then the formula

$$x = x_1 + k(x_2 - x_1)$$

expresses the coordinate x of any point on ℓ in terms of the coordinate k of the same point relative to the coordinate system with origin X_1 and unit-point X_2.

Having already proved the theorem, let us apply it to the discussion of rays and segments. In so doing, we shall discover the type of inequality condition which describes a ray or a segment in any coordinate system.

First, suppose that $x_1 < x_2$. Using the same notation as above, we express, in terms of x, the inequality $k \geq 0$ associated with the ray $\overrightarrow{X_1X_2}$.

$$k \geq 0$$

$$k(x_2 - x_1) \geq 0 \qquad \text{(since } x_2 - x_1 \text{ is positive)}$$

$$x_1 + k(x_2 - x_1) \geq x_1 \qquad \text{(by the additive property of order)}$$

or

$$x \geq x_1$$

We express, in terms of x, the inequality $0 \leq k \leq 1$ associated with the segment $\overline{X_1X_2}$.

$$0 \leq k \leq 1$$

$$0 \leq k(x_2 - x_1) \leq x_2 - x_1$$

$$x_1 \leq x_1 + k(x_2 - x_1) \leq x_2$$

or $$x_1 \leq x \leq x_2$$

We express, in terms of x, the midpoint of $\overline{X_1X_2}$. Since its coordinate in the particular system is $k = \frac{1}{2}$, we find $x = x_1 + \frac{1}{2}(x_2 - x_1) = \frac{x_1 + x_2}{2}$. In other words, the coordinate of the midpoint of a segment is the average of the coordinates of the endpoints of the segment.

A similar discussion applies in case $x_1 > x_2$.

We summarize results in the following table. On the line $\overleftrightarrow{CD}$ let C and D have respective coordinates c and d.

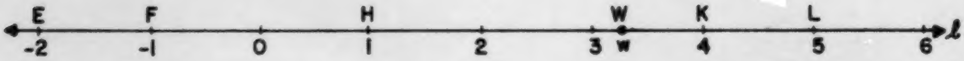
Set (or point)	In case $c < d$	In case $d < c$
$\overline{CD}$	$c \leq x \leq d$	$d \leq x \leq c$
Interior of $\overline{CD}$	$c < x < d$	$d < x < c$
$\overrightarrow{CD}$	$x \geq c$	$x \leq c$
Interior of $\overrightarrow{CD}$	$x > c$	$x < c$
Ray opposite to $\overrightarrow{CD}$	$x \leq c$	$x \geq c$
$\overrightarrow{DC}$	$x \leq d$	$x \geq d$
Midpoint of $\overline{CD}$	$x = \frac{c + d}{2}$	$x = \frac{c + d}{2}$

We recall that F is said to be between C and D if F is in the interior of $\overline{CD}$. Thus we have the following theorem which restates the betweenness property in terms of coordinates.

THEOREM 3-7. (The Betweenness-Coordinates Theorem) Let C, D, F be three points on a line ℓ and let any coordinate system on ℓ be given. The point F is between the points C and D if and only if the coordinate of F is between the coordinates of C and of D.

Of three distinct real numbers, one of them is greatest and another is least. One and only one of the numbers is between the other two. The corresponding statement for points is the following. One and only one of three distinct collinear points is between the other two.

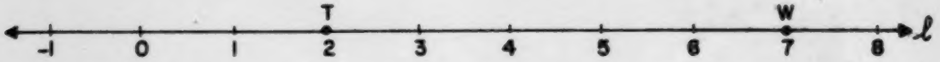
Example 1. Consider the coordinate system on the line ℓ shown in the diagram.



In the table, each of several subsets of ℓ is expressed as the set of all points on ℓ whose coordinates x satisfy a condition.

$\overline{HK}$	$1 \leq x \leq 4$
$\overrightarrow{FL}$	$x \geq -1$
$\overrightarrow{LF}$	$x \leq 5$
Midpoint of $\overline{EW}$	$x = \frac{w - 2}{2}$

Example 2. Consider the coordinate system on the line ℓ shown in the diagram.



(a) Find a point A between T and W such that $AW = \frac{1}{2} \cdot TW$.

Solution: A, the midpoint of $\overline{TW}$, is the point with coordinate $\frac{2 + 7}{2} = \frac{9}{2}$.

(b) Find a point B between T and W such that $TB = \frac{1}{3} \cdot TW$.

Solution: Since $TW = 5$, we conclude that $TB = \frac{1}{3} \cdot 5$. Thus the coordinate b of B satisfies $b - 2 = \frac{5}{3}$. Hence, $b = 2 + \frac{5}{3}$, and B is the point with coordinate $\frac{11}{3}$.

(c) If k is a positive number, find a point C on the ray opposite to $\overrightarrow{TW}$ such that $TC = k \cdot TW$.

Solution: $TC = k \cdot TW = 5 \cdot k$. Thus C is the point whose coordinate is $2 - 5k$.

(d) If k is a positive number and if V is a point whose coordinate v is greater than 2, find a point on $\overrightarrow{TW}$ such that $TD = k \cdot TV$.

Solution: $TD = k \cdot TV = k(v - 2)$. Thus, D is the point whose coordinate is $2 + k(v - 2)$.

(e) Solve (b) by a different method.

Solution: The given coordinate system on ℓ is related to the coordinate system with origin T and unit-point W by the equation $x = ak + b$. As shown in the proof of Theorem 3-6, $a = 7 - 2$ and $b = 2$. Thus, $x = 2 + 5k$. The point B, which corresponds to $k = \frac{1}{3}$, has coordinate $x = 2 + 5 \cdot \frac{1}{3} = \frac{11}{3}$.

Example 3. In a coordinate system, points A, B, Q have coordinates 7, -3, 12. In the coordinate system with origin A and unit-point B, the coordinate of Q is k. We may find the number k by using the formula in Theorem 3-6. Using $x_1 = 7$ and $x_2 = -3$, the formula is

$$x = 7 - 10k.$$

When $x = 12$, we obtain

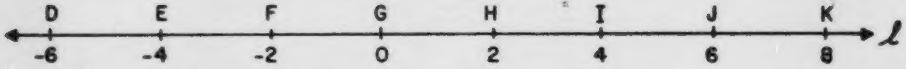
$$12 = 7 - 10k$$

$$5 = -10k$$

$$k = -\frac{1}{2}$$

Problem Set 3-10

1. Consider the coordinate system C on ℓ indicated in the diagram below. Let x be the coordinate of any point P on ℓ.



(a) Any ray or segment on ℓ is the set of all points whose coordinate x in the given coordinate system satisfy an inequality condition. Write the appropriate inequality for each of the following subsets of ℓ.

(1) $\overline{GK}$ (4) $\overrightarrow{GK}$ (7) $\overrightarrow{HJ}$ (10) $\overrightarrow{GE}$
(2) $\overline{JH}$ (5) $\overrightarrow{KG}$ (8) $\overrightarrow{HE}$ (11) $\overrightarrow{GI}$
(3) $\overline{EH}$ (6) $\overrightarrow{JH}$ (9) $\overrightarrow{EH}$ (12) $\overrightarrow{HK}$

(b) Which of the rays in (a) are the same set of points?

2. Use a coordinate system C' on ℓ in Problem 1 that assigns 0 to H and 1 to I and answer (a) and (b) of Problem 1 in terms of the C' coordinates.

3. Using the formula in the Two-Point Theorem, find an equation which expresses x in terms of k if:

(a) $x_1 = 2$, $x_2 = 6$
(b) $x_1 = -2$, $x_2 = 4$
(c) $x_1 = 3$, $x_2 = -4$
(d) $x_1 = -15$, $x_2 = 15$
(e) $x_1 = -5$, $x_2 = 0$

4. Given a coordinate system in which points A, B, C have coordinates -5, 10, 15, respectively.

(a) In the coordinate system with origin A and unit-point C find the coordinate of B, using the formula of the Two-Point Theorem. In this coordinate system does B lie in $\overline{AC}$? In $\overrightarrow{AC}$? In the ray opposite to $\overrightarrow{AC}$?

(b) In the coordinate system with origin A and unit-point B, find the coordinate of C, using the formula of the Two-Point Theorem. In this coordinate system does C lie in $\overline{AB}$? In $\overrightarrow{AB}$? In the ray opposite to $\overrightarrow{AB}$?

(c) In the coordinate system with origin B and unit-point C, find the coordinate of A. In this coordinate system does A lie in $\overline{BC}$? In $\overrightarrow{BC}$? In the ray opposite to $\overrightarrow{BC}$?

5. Find the coordinate of the midpoint of $\overline{PQ}$ if the coordinates of P and Q are, respectively:

   (a) 5 and 11

   (b) -9 and -2

   (c) $\frac{1}{2}$ and $\frac{2}{3}$

   (d) x_1 and x_2

   (e) $r + s$ and $-r$

6. The coordinate of an endpoint of a segment is 4. The coordinate of its midpoint is 7. Find the coordinate of the other endpoint.

7. If the coordinate of one endpoint of a segment is -2 and the coordinate of the midpoint is -7, find the coordinate of the other endpoint.

8. Find the coordinate of each of the trisection points of $\overline{AB}$ if the respective coordinates of A and B are:

   (a) 3 and 12

   (b) -1 and 4

   (c) x_1 and x_2

9. Consider the coordinate system C on a line containing points A, B, P. The coordinate of A is 7 and the coordinate of B is 12. Find the coordinate of P on ray $\overrightarrow{AB}$, such that

   (a) $AP = \frac{1}{4} AB$

   (b) $AP = \frac{3}{4} AB$

   (c) $PB = \frac{1}{4} AB$

10. If A and B are points on a line ℓ with respective coordinates -5 and 7 in a coordinate system C, and if A and B have coordinates 0 and 1 in a coordinate system C', then for any point P that has coordinate k in C', the coordinate x in C is $x = -5 + k(7 - (-5))$ or $x = -5 + 12k$. Using this information, complete the following table. The first row is completed.

	k	x	point on ℓ or subset of ℓ
(a)	0	-5	A
(b)			B
(c)	$\frac{1}{2}$		Midpoint of $\overline{AB}$
(d)	$\frac{1}{3}$		
(e)			P in $\overrightarrow{AB}$ such that AP = 2 AB
(f)		$-5 \leq x \leq 7$	
(g)	$k \geq 0$		
(h)			Interior of $\overline{AB}$
(i)			Interior of ray opposite to $\overrightarrow{AB}$
(j)	5		
(k)		-65	

3-11. Length.

In the preceding sections we have discussed the measurement of distance in terms of a unit-pair. We have considered the relationship between distances when different unit-pairs are used. In this section we shall suppose that a unit-pair has been chosen and is fixed. The results we obtain apply for any choice we may make.

We begin by introducing into our formal geometry a familiar word.

DEFINITION. The distance between two distinct points is called the length of the segment joining the two points.

Thus, if C and D are distinct points, the number CD is the length of the segment $\overline{CD}$.

Let $\overline{AB}$ and $\overline{CD}$ be segments. A segment is a set of points. Consequently, the statement that $\overline{AB} = \overline{CD}$ means that the sets $\overline{AB}$ and $\overline{CD}$ are the same, in other words, that $\overline{AB}$ and $\overline{CD}$ contain precisely the same points.

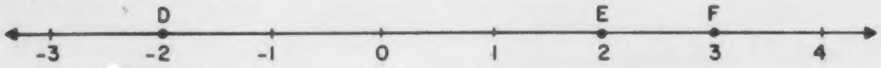
On the other hand, consider the statement $AB = CD$. This statement means that the number AB is the same as the number CD. In other words, the segment $\overline{AB}$ has the same length as the segment $\overline{CD}$. We often find useful another way of expressing the statement that two segments have the same length. This we discuss next.

In Chapter 5 we shall develop in considerable detail a notion which we call congruence. Informally, one geometric figure is congruent to another if they have the "same size and shape." We consider what this idea might mean as applied to the simple case of segments. Each segment is a set of collinear points, has two endpoints, and contains all points between its endpoints. Thus it appears that there is only one "shape" for a segment and hence that the condition "same size and shape," as applied to segments, reduces to simply "same size." Now the "size" of a segment is its length. Thus we are led to the following definition.

DEFINITION. Two segments (whether distinct or not) which have the same length are called congruent segments, and each is said to be congruent to the other.

Notation: The statement, "the segment $\overline{AB}$ is congruent to the segment $\overline{CD}$," is denoted by: $\overline{AB} \cong \overline{CD}$.

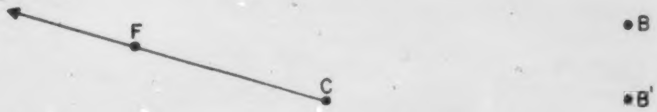
Example 1. Consider the coordinate system on the line shown in the diagram.



(a) The coordinate x of a point on $\overrightarrow{EF}$ satisfies $x \geq 2$. Is there a point G on $\overrightarrow{EF}$ such that $EG = 9$? If so, let its coordinate be g. Then $g \geq 2$ (since G belongs to $\overrightarrow{EF}$) and hence $EG = g - 2$. Thus $g - 2 = 9$, and $g = 11$. The point with coordinate 11 satisfies the requirements for G.

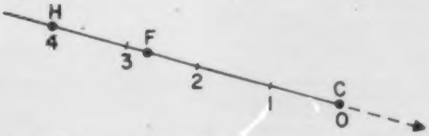
(b) The coordinate x of a point on $\overrightarrow{ED}$ satisfies $x \leq 2$. What point C on $\overrightarrow{ED}$, if any, is such that $EC = 5$? Letting the coordinate of the desired point C be called c, we note that $c \leq 2$. Thus $EC = 2 - c$. The equation $2 - c = 5$ tells us that $c = -3$. The only choice for C is the point with coordinate -3.

Example 2. Consider the unit-pair $\{B,B'\}$ and the ray $\overrightarrow{CF}$ in the diagram.



Find each point H on $\overrightarrow{CF}$ such that $CH = 4$.

Solution: On $\overrightarrow{CF}$ introduce a coordinate system relative to $\{B,B'\}$ with C as origin and with a positive coordinate for F.

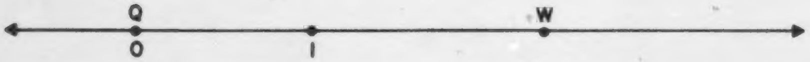


The point with coordinate 4 is the only point H on $\overrightarrow{CF}$ such that $CH = 4$.

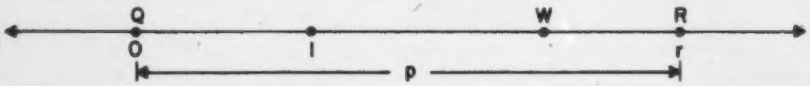
These examples suggest the following theorem.

THEOREM 3-8. (The Point Plotting Theorem) Let $\{A,A'\}$ be any unit-pair, let Q be any point, and let p be any positive number. On any ray with endpoint Q there is a unique point R such that the distance QR is p.

Proof: Let $\overrightarrow{QW}$ be any ray with endpoint Q. By Postulate 12 there is a coordinate system on $\overrightarrow{QW}$ such that Q is the origin and such that W has a positive coordinate.



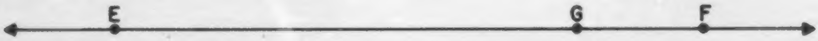
Then $\overrightarrow{QW}$ is the set of points whose coordinates in this coordinate system satisfy the inequality $x \geq 0$. The desired point R on $\overrightarrow{QW}$ must therefore have a positive coordinate, say r, and the distance $QR = r - 0$ must be p.



We choose R as the point with coordinate p and observe that this is the only possible choice. Thus our conclusion is established.

We have introduced the notion of betweenness for points. By definition, a point is between two other points if and only if it is an interior point of the segment joining them. Later we showed that a point is between two other points if and only if its coordinate is between their coordinates in a coordinate system. There is another way of characterizing betweenness, and we discuss it next.

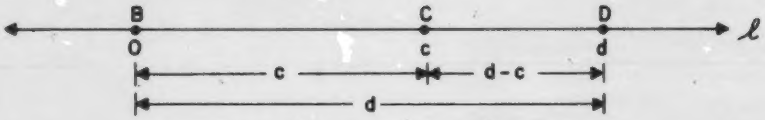
In using a yardstick to measure the length of a table in the real world, we often use an addition property. To determine the distance between the points E and F, we may select a point G between E and F.



Then the sum of the distances EG and GF is the distance EF. This idea is the basis for our next theorem.

THEOREM 3-9. (The Betweenness-Distance Theorem) Let B, C, D be points such that C is between B and D. If $\{A,A'\}$ is any unit-pair, then the distances relative to $\{A,A'\}$ satisfy the condition that $BC + CD = BD$ (or, that $BC = BD - CD$).

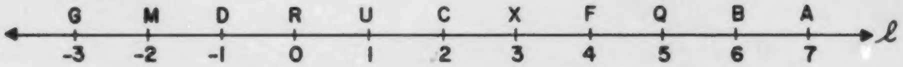
Proof: The points B, C, D belong to a line, say ℓ. By the Ruler Postulate, there is a coordinate system on ℓ relative to $\{A, A'\}$ such that B is the origin and such that D has a positive coordinate. Let d be the positive coordinate of D, and let c be the coordinate of C. The hypothesis that C is between B and D means that $0 < c < d$. The definition of a coordinate system tells us that $BC = c - 0 = c$, that $CD = d - c$, and that $BD = d$. Hence,
$BC + CD = c + (d - c) = d = BD$.



This addition property actually characterizes betweenness. That is, for any three points B, C, D which are distinct and collinear, C is between B and D if and only if $BC + CD = BD$. The portion of this result which we stated as Theorem 3-9 provides the information we need in the next few chapters, and we shall not prove the other portion at this time.

Problem Set 3-11

1. The diagram below indicates the coordinates which are assigned to various points on line ℓ by a coordinate system.

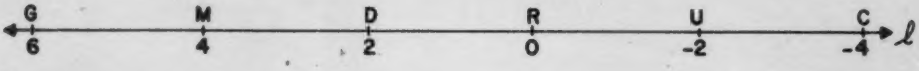


   (a) Find the length of $\overline{XB}$, $\overline{AF}$, $\overline{GD}$, $\overline{FG}$.
   (b) Find the length of $\overline{DB}$; find the distance between point D and B; find DB; find BD. Should the answer in each case be the same? Explain.

2. Indicate in each of the following whether the statement is true or false. Explain your answer in each case. (The statements refer to the diagram in Problem 1.)

   (a) $UR = 1$
   (b) the length of $\overline{UX}$ is 3
   (c) the length of $\overline{XU}$ is 2
   (d) $\overline{AD} = \overline{MB}$
   (e) $AD = MB$
   (f) $\overline{AD} \cong \overline{GR}$

3. The diagram below indicates the coordinates which have been assigned to various points on line ℓ by a coordinate system.



In each of the following, if the statement is meaningful, indicate whether it is true or false. If a statement is not meaningful, write "not meaningful" as your answer.

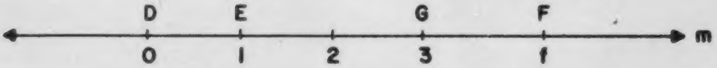
(a) $GD = RC$	(f) $\overline{GR} \cong \overline{DC}$
(b) $\overline{GM} = MD$	(g) M is the midpoint of GD
(c) $\overline{GM} \cong MD$	(h) M is between D and C
(d) $\overline{RM} \cong \overline{RC}$	(i) $DR + \overline{RC} = DC$
(e) $\overline{DG} \cong \overline{RU}$	(j) $\overline{MU} \cong \overline{MU}$

4. If the points M, S, and T are collinear in that order, express MT in terms of MS and ST. Justify your answer.

5. Points P, Q, and R are collinear in that order and the coordinates -5, 3 and -2 are assigned to them in some order. Could the coordinate of Q be 3 ? Why?

6. The points A, H and J are collinear in that order. If $AJ = 12$, and $HJ = 7$, find AH.

7. The diagram below indicates coordinates assigned to the points D, E, G and F on line m by a coordinate system.



(a) Is there a point on $\overrightarrow{DE}$ whose coordinate is 9; a point whose coordinate is -5 ? Justify your answers.

(b) Point F belongs to $\overrightarrow{DE}$. Find the coordinate of F in each of the following. (In any case in which F is not determined by the given information, give all possible answers.)

(1) $DF = 7\ DE$ (5) $EF = 5\ EG$

(2) $DF = \frac{1}{3}\ DE$ (6) $EF = \frac{1}{2}\ EG$

(3) $DF = 5\ EG$ (7) $EF = 2\ DG$

(4) $EF = 7\ DE$ (8) $EF = k \cdot EG$

8. Let A, B, P be points on $\overleftrightarrow{AB}$ with coordinates -2, 5, p, respectively. Let k be a positive number and let $AP = k \cdot AB$.

(a) (1) find p if $k = 2$
 (2) find p if $k = 1$
 (3) find p if $k = \frac{1}{2}$

(b) (1) find k if $p = 3$
 (2) find k if $p = 19$

(c) If R is on $\overleftrightarrow{AB}$, and $AR = 2$, what is the coordinate of R ?

9. The collinear points E, F, and G have coordinates 0, 18 and x, respectively. Express each of the following in terms of x :

(a) (1) EG if $0 < x < 18$
 (2) GF if $0 < x < 18$
 (3) EG if $x > 18$
 (4) GF if $x > 18$
 (5) GF if $x < 0$

(b) For each of the following statements, what restriction would you have to place on x, the coordinate of G, so that the statement would be true?
 (1) $EG + GF = EF$
 (2) $EF + FG = EG$
 (3) $GE + EF = GF$

10. If P, Q and R are collinear in that order, show that $PQ < PR$ and $QR < PR$.

3-12. Summary.

After a brief review of the pertinent properties of the real number system, we stressed four major topics in Chapter 3.

We introduced the notion of a distance between any two points in space. Postulate 10 states that there is a number measuring the distance between two given points and that this number is unique relative to a chosen unit-pair. Postulates 11 and 13 describe how the number is affected (if at all) by replacing the unit-pair with another unit-pair.

Our work with coordinate geometry began when we described a coordinate system on a line. A coordinate system is a one-to-one correspondence between the line and the set of all real numbers which relates distances between points on the line and differences between numbers. The Ruler Postulate states that every line has a coordinate system. Furthermore, it guarantees that there is only one coordinate system such that a given point is origin and another given point has a positive coordinate. The Origin and Unit-Point Theorem allows us to assign the respective specific coordinates 0 and 1 to any two distinct points on the line, if we are willing to measure distances relative to that same pair of points. Two later theorems (the Two Coordinate Systems Theorem and the Two-Point Theorem) describe how two different coordinate systems on a line may be related.

Several important types of subsets of a line are the ray, the segment, the interior of a ray, the interior of a segment. Each of these types can be efficiently described by means of a coordinate system, since each of them is the set of all points on the line whose coordinates satisfy a condition expressed by one or more inequalities.

We discussed the concept of betweenness and noted several different properties. Let F, C, D be three distinct collinear points and let a coordinate system on the line containing them be given. Then each of the following statements is equivalent to each of the others:

(a) F is between C and D;

(b) F is an interior point of $\overline{CD}$;

(c) the coordinate of F is between the coordinates of C and D.

(d) CF + FD = CD.

In the next chapter we shall talk about rays which are not collinear as we begin our study of angles.

VOCABULARY LIST

unit-pair

measure of distance

coordinate system;
  origin
  unit-point

coordinate

ray

segment

interior (of rays and segments)

betweenness (for points)

midpoint (of a segment)

length (of a segment)

congruent segments

Review Problems

1. Given the following sets:

   $V =$ set of real numbers
   $R =$ set of rational numbers
   $I =$ set of irrational numbers
   $J =$ set of integers
   $N =$ set of natural numbers
   $\emptyset =$ empty set
   ($\cap$ means intersection
   $\cup$ means union)

   Fill in the blanks with one of the six symbols above.

   (a) $R \cup$ _____ $= V$
   (b) $\{1,2,3,\dots\} =$ _____.
   (c) J is a subset of _____.
   (d) $R \cap I =$ _____.
   (e) a subset of J is _____.

2. Make each sentence in Column I a true statement by filling in the blank with a phrase from Column II.

   Column I

   (a) If $a > b$, then $a - b$ is _____.
   (b) If $0 < k$, and $k^2 < 4$, then k is _____.
   (c) If $a < b$, then $a - b$ is _____.

   Column II

   (1) less than 2
   (2) positive
   (3) negative
   (4) positive number less than 2.
   (5) greater than 2.

3. Let r and s be nonzero real numbers such that $r > s$. For each of the following, indicate whether (T) the statement is true, or (F) the statement is false, or (N) the given information in insufficient to determine whether the statement is true or false.

   (a) $s < r$
   (b) $r - s > 0$
   (c) $r - 2 < s - 2$
   (d) $\frac{r}{s} > 1$
   (e) $r^2 > s^2$

4. Follow the instructions of Problem 3 for the following:

(a) $\frac{1}{r} > \frac{1}{s}$. (c) $r^3 > s^3$.

(b) $\frac{1}{2}s < \frac{1}{2}r$. (d) $1 - r < 1 - s$.

5. Find the solution set for each of the following inequalities:

(a) $-5x > 15$. (d) $x - 1 > 3x + 2$.

(b) $0 < 7 - 3x$. (e) $2x + 1.5 \geq x + 1$.

(c) $2 - x < 16$.

6. (a) Plot the solution set for Problem 5(a).

(b) Plot the solution set for Problem 5(e).

7. On a line let a coordinate system be given. Find the distance between points having the following coordinates.

(a) 0 and 8. (f) $3b$ and $-4b$; $b < 0$

(b) 0 and -8. (g) $(a - b)$ and $(a + b)$; $b \geq 0$.

(c) -5 and -2. (h) y_1 and y_2

(d) -5 and -14. (i) $(a - b)$ and $(a + b)$; $b < 0$.

(e) 3.3 and -7.8 .

8. The first numbering of the points below the line is given by a coordinate system. Which of the other numberings are not given by coordinate systems?

	-3	-2	-1	0	1	2	3	4	5	6	7
(a)	-7	-6	-5	-4	-3	-2	-1	0	1	2	3
(b)	0	1	2	3	4	5	4	3	2	1	0
(c)	11	12	13	14	15	16	17	18	19	20	21
(d)	-11	-12	-13	-14	-15	-16	-17	-18	-19	-20	-21
(e)	-3	-2	1	0	-1	2	3	4	5	6	7

9. In each part of this problem, consider the set of all points on a line whose coordinates x satisfy the condition given. Which of these sets is a ray? a point? a line? a segment?

(a) $x < 3$	(f) $-3 \leq x \leq 3$
(b) $x = 1$	(g) $2 < x \leq 2$
(c) $x > 2$	(h) $x \geq 0$ or $x < 0$
(d) $x \leq 1$	(i) x is between 3 and 4.
(e) $x = -3$	(j) $0 \leq x \leq 1$ or $x \geq 1$

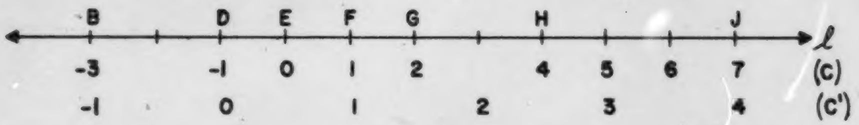
10. Find the coordinate of the midpoint of a segment if the coordinates of its endpoints are:

(a) -7 and 10
(b) 3 and 10
(c) r and s

11. The coordinate of the midpoint of a segment $\overline{CD}$ is 4. Find the coordinate of point C if the coordinate of point D is:

(a) -3
(b) 9
(c) 3 times the coordinate of C.

12. In this problem C is a coordinate system on ℓ with respect to [E,F] and C' a coordinate system on ℓ with respect to [D,F]. Find each of the numbers indicated.



(a) $\dfrac{\text{FJ (relative to [E,F])}}{\text{FJ (relative to [D,F])}}$; $\dfrac{\text{DH (relative to [E,F])}}{\text{DH (relative to [D,F])}}$

(b) $\dfrac{\text{FJ (relative to [E,F])}}{\text{DH (relative to [E,F])}}$; $\dfrac{\text{FJ (relative to [D,F])}}{\text{DH (relative to [D,F])}}$

(c) DF (relative to [E,F])

(d) Compare your answers in (a). Which postulate or theorem predicted this result?

(e) Compare your answers in (b). Which postulate or theorem predicted this result?

13. On a line let a coordinate system be given such that the points P, Q, R, S, T have coordinates 2, -1, 0, -3, 4, respectively.

Indicate whether each of the following statements is meaningful or not meaningful. If meaningful, indicate whether the statement is true or false.

(a) $\overline{SQ} \cong \overline{QR}$

(b) $SQ = \overline{PT}$

(c) $RP =$ (length of $\overline{SQ}$)

(d) $QS + SR = QR$

(e) $\overline{QP} = \overline{QP}$.

14. Let $\overleftrightarrow{RW}$ be a line to which a coordinate system has been assigned such that R and W have the coordinates 0 and 1 respectively.

(a) Is there a point on $\overleftrightarrow{RW}$ which has the coordinate 17 ? Justify your answer.

(b) Is there a point on $\overleftrightarrow{RW}$ which has the coordinate -17?

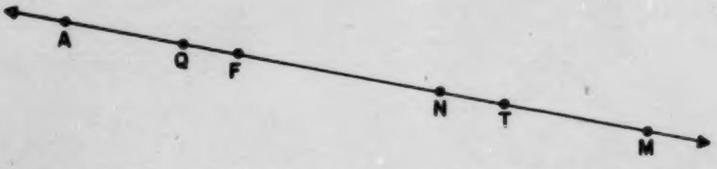
(c) Find the coordinate of point P on $\overrightarrow{RW}$ such that

(1) $RP = 12$
(2) $RP = 8 \cdot RW$
(3) $WP = 5 \cdot RW$
(4) $WP = k \cdot RW$

(d) In each of the four parts of (c), is the information sufficient to determine P so that only one answer is possible?

(e) How would your answers in part (c) be affected if you were not given that P is on $\overrightarrow{RW}$ but instead were given that P is on $\overleftrightarrow{RW}$?

15. The points A, Q, F, N, T, M are collinear in that order.



For each of the following statements, determine whether it is true or false.

(a) $\overrightarrow{QF} = \overrightarrow{QN}$.

(b) $\overline{FN}$ is a subset of $\overleftrightarrow{AQ}$.

(c) M and N belong to opposite rays which are contained in $\overleftrightarrow{AF}$ and have common endpoint T.

(d) $\overrightarrow{NM}$ is a subset of $\overrightarrow{FT}$.

(e) $\overrightarrow{NT}$ is a subset of $\overline{FM}$.

(f) The intersection of $\overrightarrow{FQ}$ and $\overrightarrow{NT}$ is empty.

(g) The intersection of $\overrightarrow{FQ}$ and $\overrightarrow{TN}$ is $\overrightarrow{FQ}$.

(h) The union of $\overline{NF}$ and $\overrightarrow{FA}$ is $\overrightarrow{NA}$.

(i) The union of $\overline{MQ}$ and $\overrightarrow{MA}$ is $\overrightarrow{MA}$.

(j) The intersection of $\overline{QN}$ and $\overline{FM}$ is $\overline{FN}$.

16. (a) Draw two segments $\overline{AB}$ and $\overline{CD}$ for which the intersection of $\overline{AB}$ and $\overline{CD}$ is the empty set but the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ is one point.

(b) Draw two segments $\overline{PQ}$ and $\overline{RS}$ for which the intersection of $\overline{PQ}$ and $\overline{RS}$ is the empty set but $\overleftrightarrow{PQ} = \overleftrightarrow{RS}$.

(c) Draw a line. On the line label three points A, B, C with B between A and C.

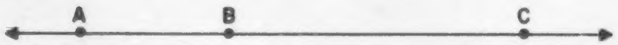
   (1) What is the intersection of $\overline{AB}$ and $\overline{BC}$? of $\overline{AC}$ and $\overline{BC}$?

   (2) What is the union of $\overline{AB}$ and $\overline{BC}$? of $\overline{AB}$ and $\overline{AC}$?

(3) What is the intersection of the interior of $\overline{AB}$ and the interior of $\overline{CB}$?

(4) What is the intersection of the interior of $\overrightarrow{AB}$ and the interior of $\overrightarrow{CB}$?

17.



(a) Write an equation that describes the relative positions which these three collinear points appear to have.

(b) Under what additional condition would B be the midpoint of $\overline{AC}$?

18. Below are given one hypothesis and three conclusions. In each case, indicate the definition or theorem which justifies the statement that the conclusion follows from the hypothesis.

If three collinear points R, S, T have respective coordinates 4, 5, 8,

(a) then S is between R and T because $4 < 5$ and $5 < 8$.

(b) then R cannot be between S and T.

(c) then $RS + ST = RT$.

19. Let P, Q, R be three points on a line such that PR (in cm.) = 30, QR (in m.) = 0.4, and QP (in mm.) = 100. Which point is between the other two? Explain your answer.

20. Let A and B be two points. In a certain coordinate system on $\overleftrightarrow{AB}$, the segment $\overline{AB}$ is the set of all points whose coordinates x satisfy the condition $3 \leq x \leq 10$. The coordinate of A is less than the coordinate of B.

(a) What are the coordinates of the endpoints of $\overline{AB}$?

(b) What is the coordinate of the endpoint of $\overrightarrow{AB}$?

(c) What is the coordinate of the endpoint of $\overrightarrow{BA}$?

(d) What is the coordinate of the endpoint of the ray opposite to $\overrightarrow{BA}$?

21. On a line, consider points A, B with coordinates 4 and -5, respectively. Let x be the coordinate of a point X on $\overleftrightarrow{AB}$. If $x - 4 > 0$, which of the rays, $\overrightarrow{AB}$ or $\overrightarrow{BA}$, contains X ?

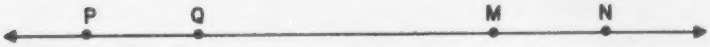
22. Consider a line and a coordinate system on the line. Let A, B, C be the points on the line with respective coordinates -3, 7, 31. Let x be the coordinate of a point X on $\overrightarrow{AB}$, let y be the coordinate of a point Y on $\overrightarrow{AC}$, and let z be the coordinate of a point Z on $\overrightarrow{CB}$. Use inequalities to show all possible values:

(a) of x ;
(b) of y ;
(c) of z.

23. Let A, B, P, be three distinct points on a line with coordinates a, b, p, respectively, and $a < p < b$.

(a) If $PA = PB$, express p in terms of a, b.
(b) If $PA = \frac{1}{3} PB$, express p in terms of a, b.
(c) If $\frac{PA}{AB} = \frac{2}{5}$, express p in terms of a, b.
(d) If $\frac{PA}{AB} = k$, express p in terms of a, b.
(e) In (d) above, can k be negative? Can $k \geq 1$? Can $k = 0$? Explain why.

24.



Referring to the above figure, explain the meaning of the following symbols:

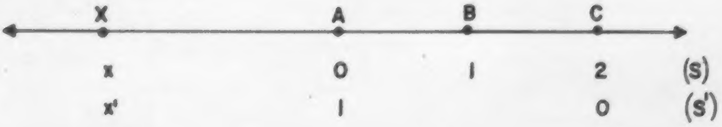
(a) $PQ = MN$.
(b) $\overline{PQ} \cong \overline{MN}$.
(c) $\overleftrightarrow{QM} = \overleftrightarrow{NP}$.
(d) $QN > QM$.
(e) $\overrightarrow{QN} = \overrightarrow{QM}$.

25. Let B, A, C be three points, collinear in that order.
(a) Is $\overrightarrow{AB}$ a subset of $\overleftrightarrow{AB}$?
(b) What is the union of $\overrightarrow{AB}$ and $\overrightarrow{AC}$? the intersection of $\overrightarrow{AB}$ and $\overrightarrow{AC}$?

(c) What is the union of $\overrightarrow{AB}$ and $\overleftrightarrow{AB}$? the intersection of $\overrightarrow{AB}$ and $\overleftarrow{AB}$?

(d) What is the intersection of $\overline{AB}$ and $\overrightarrow{AB}$? the union of $\overline{AB}$ and $\overrightarrow{AB}$?

26. Given two coordinate systems S and S' suggested by the following diagram:



(a) Express x' (the coordinate of X in system S') in terms of x (the coordinate of X in system S.)

(b) Determine the coordinate of B in S'.

27. Suppose that points D, E, F, G, H lie on a line. Data from four different coordinate systems on this line are tabulated below. The column headed "Relationship" gives the relationship between the coordinate of each point in the system and the coordinate x of the same point in the first system. Fill in the missing entries in the table.

(Suggestion: Complete the top row in the table before starting the bottom row. In the bottom row find the missing formula in the Relationship column before finding the other missing entries in the row.)

Coordinate System	Relationship	Coordinate of					
		D	E	F	G	H	I
First	x	0	3		-1		
Second	x' = 6x						-12
Third	x'' = x - 3			-1		2	
Fourth	x''' = _____	2				-8	

28. On a line ℓ the coordinates of points A and B are x_1 and x_2, respectively, and x is the coordinate of any point X on ℓ. Indicate the value or values of k in the equation $x = x_1 + k(x_2 - x_1)$ which would restrict X to the following:

(a) $\overline{AB}$
(b) $\overrightarrow{AB}$
(c) ray opposite $\overrightarrow{AB}$
(d) $\overrightarrow{BA}$
(e) $\overleftrightarrow{AB}$

29. When we write AB + BC = AC, where A, B, C are points on a line, does this imply that B is between A and C ? Explain your answer.

30. On a line ℓ are given points R, S, and T whose coordinates are 3, 0, -5, respectively, in coordinate system C and -3, s, 13 is coordinate system C'.

(a) Give an equation in the form $x' = ax + b$, to show the relationship between coordinate x' in C' of any point and its corresponding coordinate x in C.

(b) Compute s.

31. On a line ℓ are given points A, B, and P whose coordinates are 0, 1, k, respectively, in coordinate system C and x_1, x_2, and x' in coordinate system C'. Represent x' in terms of x_1, x_2 if k is

(a) 0
(b) 1
(c) $\frac{1}{2}$
(d) -2

Chapter 4

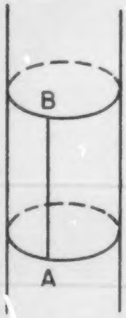
ANGLES

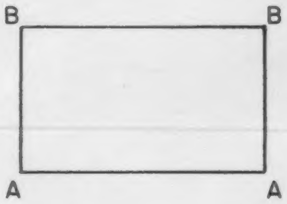
4-1. Introduction.

In Chapter 3 we introduced a measure of "how far apart" two points are, we described coordinate systems on a line, and we discussed the concept of betweenness for three distinct collinear points. In the present chapter we introduce a measure of "how far apart" two concurrent rays are, we describe ray-coordinate systems in a plane, and we discuss the concept of betweenness for rays. As a prelude to our work on angles in this chapter, we need to develop more fully ideas concerning separation for which our work in the preceding chapter prepared the background.

4-2. Separation.

In our discussion of opposite rays in Chapter 3, we saw that a point on a line separates the line into two parts. Extending this idea, it is natural to ask if a line in a plane separates the plane into two parts and to inquire whether a plane separates space into two parts. On the basis of your past experience, you would probably expect the answers to these questions to be Yes. But then you might think of a cylindrical surface, like that suggested by rolling up a sheet of paper.





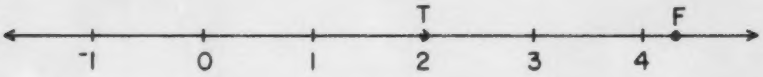
For a surface like this, a straight line, such as $\overleftrightarrow{AB}$, does not separate it into two parts! Could this also happen for a plane? Can we prove that it cannot happen for a plane? Curiously, with the postulates that we have so far agreed upon, it is impossible to prove this. Although our discussion of opposite rays suggests how a point on a line separates the line, we cannot prove that a line in a plane separates the plane nor that a plane separates space. Consequently, if we want planes and space to have this type of "separable" property, as lines do, we must adopt additional postulates.

To facilitate the phrasing of the new postulates that we must add to our system, it is convenient to introduce the idea of a convex set of points.

> DEFINITION. A set containing more than one point is said to be a convex set if and only if, for every two points of the set, the segment joining the points is contained in the set. Every set of points which contains no more than one point is also said to be a convex set.

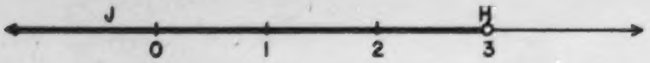
In symbols, the condition that a set $\mathcal{S}$ of points be convex is the following: if P and Q are any two distinct points in $\mathcal{S}$, then $\overline{PQ}$ is a subset of $\mathcal{S}$.

Example 1. Consider the ray $\overrightarrow{TF}$ in the diagram.



The ray consists of all points whose coordinates x satisfy $x \geq 2$. Suppose that P and Q are distinct points in $\overrightarrow{TF}$. Each of their coordinates is equal to or greater than 2. Thus any point belonging to the segment $\overline{PQ}$ has a coordinate which is equal to or greater than 2. In other words, $\overline{PQ}$ is a subset of $\overrightarrow{TF}$. The ray $\overrightarrow{TF}$ is a convex set.

Example 2. In the following diagram let $\mathcal{S}$ be the interior of the ray $\overrightarrow{HJ}$.

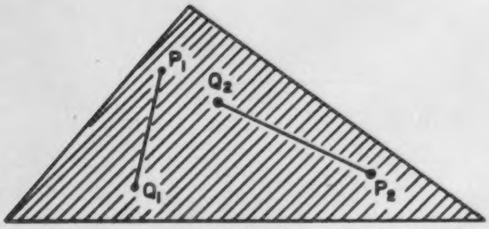


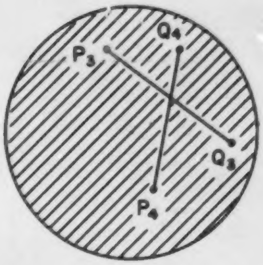
The set $\mathcal{S}$ consists of all points whose coordinates are less than 3 . Suppose that P and Q are distinct points in $\mathcal{S}$.

Each of their coordinates is less than 3 . Any point in $\overline{PQ}$ therefore has a coordinate less than 3 and consequently belongs to $\mathcal{S}$. The set $\mathcal{S}$ is a convex set.

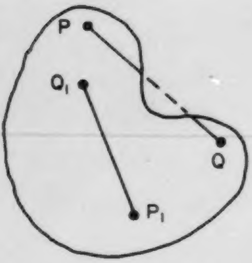
Example 1 illustrates the fact that every ray is a convex set. The interior of a ray is a set which is often called a halfline. Example 2 illustrates the fact that every halfline is a convex set.

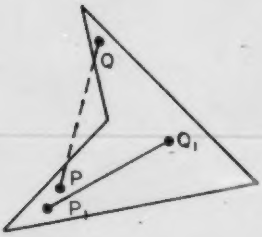
A convex set does not need to be a set of collinear points. For example, the diagrams below suggest that the interior of a triangle is a convex set, and that the interior of a circle is a convex set. Each diagram shows two possible choices (distinguished by subscripts) for points P and Q belonging to the set. In each case, note that the set contains the segment joining the points.

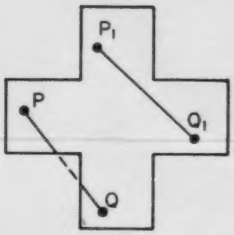




On the other hand, there are many regions which are not convex sets. Each of the three pictures below shows a set which is not convex. Note that for some possible choices of two points in the set, say P_1 and Q_1, the segment $\overline{P_1Q_1}$ may be contained in the set. Nevertheless each set contains at least one pair of points, say P and Q, such that the set does not contain all points of the segment $\overline{PQ}$.







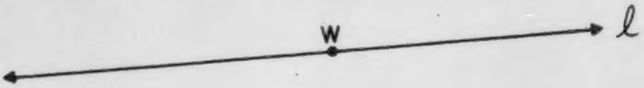
THEOREM 4-1. The intersection of any two convex sets of points is a convex set.

Proof: Let the given convex sets be called S and T. If their intersection has no more than one point, it is convex, by definition. Suppose then the intersection contains more than one point. Consider any two distinct points, say P and Q, which belong to the intersection of S and T. We must show that $\overline{PQ}$ belongs to the intersection. Each of the points P and Q belongs to the intersection and hence is a member of S; therefore, since S is convex, S contains $\overline{PQ}$. Similarly each of the points P and Q belongs to the intersection and hence is a member of T; therefore, since T is convex, T contains $\overline{PQ}$. Since we have deduced that $\overline{PQ}$ is a subset of S and also a subset of T, we conclude that $\overline{PQ}$ is contained in the intersection of S and T. Thus the intersection of S and T is a convex set.

Example 3. Let A and B be distinct points. The segment $\overline{AB}$ is the intersection of the rays $\overrightarrow{AB}$ and $\overrightarrow{BA}$. Since each of the rays is a convex set, the segment $\overline{AB}$ is a convex set.

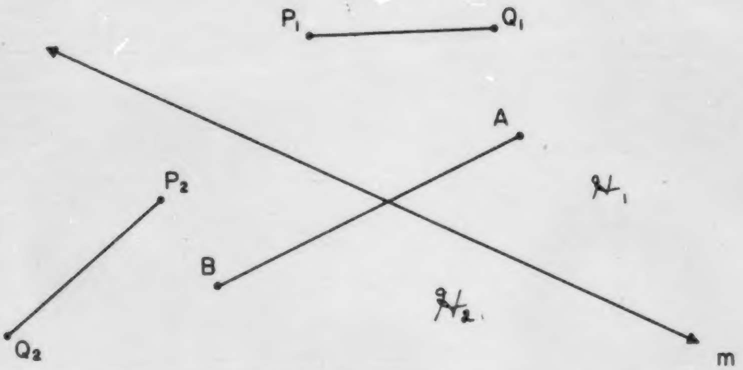
Example 4. Let the points E, F, G, be collinear in that order. The opposite rays $\overrightarrow{FE}$ and $\overrightarrow{FG}$ intersect in the single point F and this intersection is a convex set.

Example 5. Let W be a point on a line ℓ.



Two rays on ℓ are determined by W. Each ray determines a halfline. The two halflines, which appear to be "on opposite sides of" W, do not intersect. Furthermore, W is in the interior of any segment whose endpoints are in different halflines on ℓ determined by W.

Now consider a line m in a plane as shown in the diagram. The line seems to separate the plane into two regions, say $\mathcal{H}_1$ and $\mathcal{H}_2$, which form "opposite sides" of the line. Each of these two regions seems to be a convex set.



For example, the segment joining the points P_1 and Q_1 in $\mathcal{H}_1$ is contained in $\mathcal{H}_1$. The segment joining P_2 and Q_2, which are points in $\mathcal{H}_2$, is contained in $\mathcal{H}_2$. However it appears that, although any segment determined by two points on the "same side" of m does not intersect m, every segment joining points on "opposite sides" of m does intersect m. For instance, the segment with endpoint A in $\mathcal{H}_1$ and endpoint B in $\mathcal{H}_2$ intersects m. With these observations to guide us, we now state our next postulate.

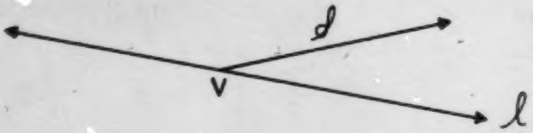
Postulate 14. (The Plane Separation Postulate) For any plane and any line contained in the plane, the points of the plane which do not lie on the line form two sets such that

(1) each of the two sets is convex, and
(2) every segment which joins a point of one of the sets and a point of the other intersects the given line.

DEFINITIONS. Each of the two convex sets determined by a given line in a given plane, according to Postulate 14, is called a *halfplane*. The line is called the *edge* of each halfplane. The line is said to *separate* the plane into the two halfplanes. Each of the halfplanes is called a *side* of the line, and the two halfplanes are said to be *opposite sides* of the line.

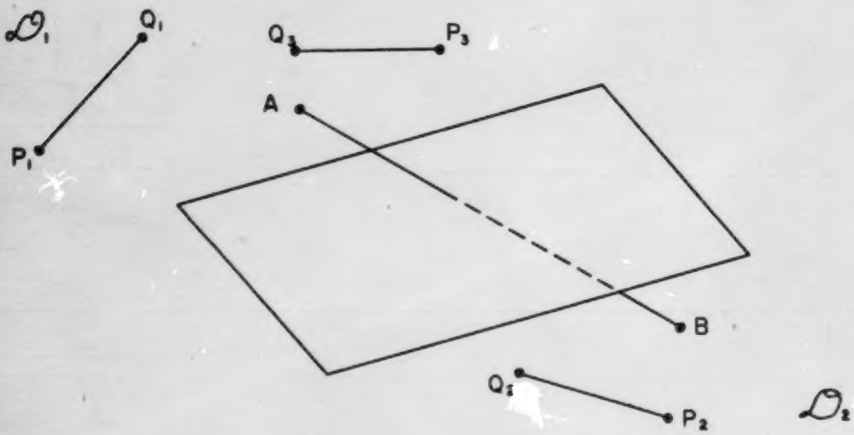
THEOREM 4-2. If the intersection of a line and a ray is the endpoint of the ray, then the interior of the ray is contained in one of the halfplanes whose edge is the given line.

Proof: Suppose the given line ℓ and the given ray intersect at the endpoint V of the ray. Let $\mathcal{I}$ be the interior of the ray.



Since ℓ and the ray have only one point of intersection, no point of $\mathcal{I}$ belongs to ℓ. If there were two points of $\mathcal{I}$ on opposite sides of ℓ, then a point between them, say M, would be on ℓ, by Postulate 14. On the other hand, M would be in $\mathcal{I}$, since $\mathcal{I}$ is a convex set. This contradiction shows that $\mathcal{I}$ cannot contain points in different halfplanes with edge ℓ. Thus $\mathcal{I}$ is entirely contained in one of the halfplanes.

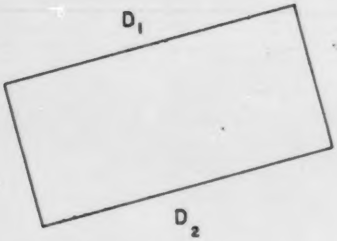
Just as a line separates a plane which contains it, so a plane appears to separate space into two convex sets, such as $\mathcal{D}_1$ and $\mathcal{D}_2$ in the diagram, each of which we may call a halfspace.

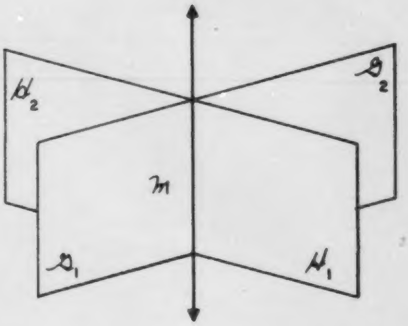


To ensure that this is actually the case in our geometry, we agree on another postulate.

> Postulate 15. For any plane, the points of space which do not lie on the plane form two sets such that
>
> (1) each of the two sets is convex, and
> (2) every segment which joins a point of one of the sets and a point of the other intersects the given plane.

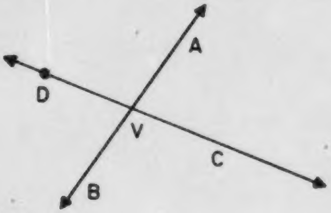
We should note that whereas a given plane determines exactly two halfspaces, every line is the edge of infinitely many halfplanes, because there are infinitely many planes containing a given line and each of these is separated into two halfplanes by the line.





Problem Set 4-2

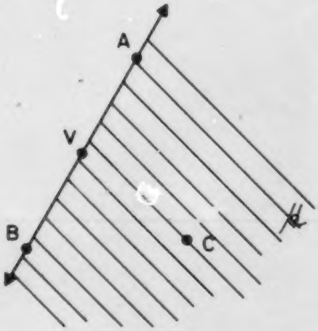
1. Let A, B, C, D, V be five points such that the two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at V. Let $\mathcal{H}$ be the halfplane with edge $\overleftrightarrow{AB}$ and containing C ; let $\mathcal{G}$ be the halfplane with edge $\overleftrightarrow{CD}$ and containing A .



   (a) Explain why $\mathcal{H}$ and $\mathcal{G}$ are subsets of the same plane.

   (b) Draw your own diagram showing the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ and the points A, B, C, D, V .

   (c) On your diagram represent the halfplane $\mathcal{H}$ by marking such as



   (d) Similarly represent $\mathcal{G}$, but use markings in a different direction in order to emphasize the distinction between $\mathcal{G}$ and $\mathcal{H}$.

   (e) How is the intersection of $\mathcal{H}$ and $\mathcal{G}$ marked?

   (f) Why is the intersection of $\mathcal{H}$ and $\mathcal{G}$ a convex set?

2. Complete the following proof that the intersection of two coplanar halfplanes is a convex set.

   Let $\mathcal{H}$ and $\mathcal{G}$ be two coplanar halfplanes. Both $\mathcal{H}$ and $\mathcal{G}$ are _____ sets because by Postulate 14 every _____ is a convex set. Then by Theorem 4-1, their __________ is also a _____ set.

3. Complete the following proof that every line is a convex set.

   Let ℓ be any line. Let P and Q be any two distinct _____ of ℓ . Then any point in the interior of $\overline{PQ}$ is in _____ , because $\overline{PQ}$ is a _____ of $\overleftrightarrow{PQ}$. Therefore, by definition, ℓ is a convex set.

4. Complete the following proof that every plane is a convex set.

   Let m be any plane. Let P and Q be any two distinct points of m . By Postulate _____, every point of $\overleftrightarrow{PQ}$ is in m . Since $\overline{PQ}$ is a subset of $\overleftrightarrow{PQ}$, it follows that $\overline{PQ}$ is a _____ of m . Therefore m is a convex set.

5. Suppose the intersection of $\overleftrightarrow{RS}$ and $\overline{AB}$ is A . Prove that all points of $\overline{AB}$ except A lie in the same halfplane with edge $\overleftrightarrow{RS}$. The following questions should help you write the proof.

   (a) Let P be a point between A and B . Is P contained in $\overleftrightarrow{RS}$? Why?
   (b) What is the intersection of $\overline{PB}$ and $\overleftrightarrow{RS}$? Why?
   (c) Can P be on the opposite side of $\overleftrightarrow{RS}$ from B ? Why?
   (d) Must P be on the same side of $\overleftrightarrow{RS}$ as B ? Why?
   (e) Do all points on $\overline{AB}$ except A lie in the same halfplane with edge $\overleftrightarrow{RS}$? Why?

6. Suppose the intersection of $\overrightarrow{RS}$ and $\overline{AB}$ is A and A is an interior point of $\overrightarrow{RS}$. To prove that all points of $\overline{AB}$ except A lie in the same halfplane with edge $\overrightarrow{RS}$, we may think of $\overrightarrow{RS}$ as a subset of _____ and apply the proof of the statement in Problem 5.

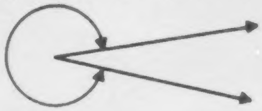
7. Consider the variation of Problem 2 in which the halfplanes are not coplanar.
   - (a) Is the intersection of the two halfplanes a convex set? Why?
   - (b) The intersection of two noncoplanar halfplanes is a convex set of one of several types. For example, the intersection may be a line. List some others of these types.

4-3. The Concept of Angle.

The concept of an angle is very basic in the study of geometry. In this section we discuss some of the interpretations given to the notion of an angle and select the most appropriate one for our development of geometry.

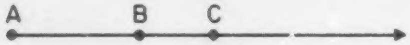
Some of the most familiar ways of thinking about an "angle" are: (1) as a particular set of points, specifically as the union of two concurrent rays; (2) as an ordered pair of rays, distinguished by the names "initial" and "terminal"; and (3) as a rotation of a ray about its endpoint from one position to another. In our geometry the angles we discuss are often angles of a triangle or perhaps angles associated with a circle or other geometric figure. Since each of these figures is regarded as a set of points, we prefer to think of an angle as being a set of points. The other interpretations of an angle have considerable importance in other branches of mathematics and science.

In some situations there may be a need to distinguish between an angle (in the sense of the smaller of two pieces of pie) and the "other angle," called a reflex angle, which has the same rays for its sides. A reflex angle is usually indicated in a diagram by a double-headed arrow.

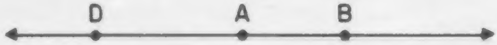


However our development of geometry does not require this type of distinction, and so we shall not use reflex angles.

The ideas of a "straight angle" and a "zero angle" fit our description of an angle as the union of two concurrent rays. A zero angle can be thought of as the union of two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in the case in which $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are the same.



Similarly the union of two opposite rays $\overrightarrow{AB}$ and $\overrightarrow{AD}$ can be thought of as a straight angle.



The admission of zero angles and straight angles into our development would involve a number of complications. For instance, several important theorems about angles do not hold for zero angles or straight angles, and the wording of these theorems, if revised to take care of the special cases, would be awkward. This is particularly true when in a later section, we describe the interior of an angle. As a result of our intuitive notion of the interior of an angle, we should probably agree that (1) for a straight angle, it is not sensible to define an interior, and (2) for a zero angle, the interior is simply the empty set. The exceptional situation in both these cases is one source of complication which would result from our introduction of straight angles and zero angles.

Another difficulty arises with respect to the identification of the vertex of a straight angle. Suppose we consider an angle (properly) as a set of points. Then the union of two opposite rays, that is, a straight angle, is simply a line, and no point is distinguishable from any other point of the set. Thus the vertex cannot be identified among the points of the set.

Since we shall be concerned primarily with figures such as triangles and other polygons, in which neither zero angles nor straight angles occur, we have nothing to gain by using these angles. We prefer to avoid all the complications which we have been discussing, and we do so by insisting that an angle be a union of two concurrent rays which are furthermore not collinear.

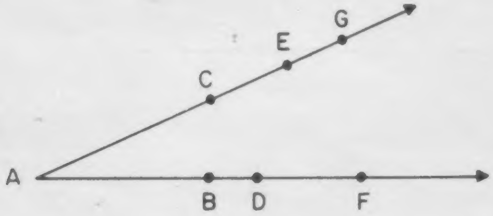
We summarize this section by stating the definition of an angle which is most appropriate for us.

> DEFINITIONS. An angle is the union of two rays which have a common endpoint but do not lie in the same line. Each of the two rays is called a side of the angle. The common endpoint of the two rays is called the vertex of the angle.

We note that the definition of an angle may be rephrased in other terminology as follows: an angle is the union of two concurrent noncollinear rays.

> Notation. We often denote the angle formed by the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ by the symbol $\angle BAC$.

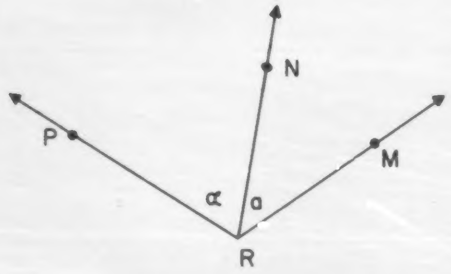
When using this notation it is important to remember that the middle letter is always the letter which names the vertex of the angle. We should also understand clearly that there are many "three-letter" symbols which can be used to name a given angle. For, as we observed in the preceding chapter, a ray can be named equally well by its endpoint and any other one of its points.



Hence in the figure, $\overrightarrow{AB} = \overrightarrow{AD} = \overrightarrow{AF}$ and $\overrightarrow{AC} = \overrightarrow{AE} = \overrightarrow{AG}$, and therefore

$$\angle BAC \text{ , } \angle DAE \text{ , } \angle FAG \text{ , } \angle EAB \text{ , }$$

are all names for the same angle, that is, the same set of points. (Give some other names for this angle.) Sometimes, where there is no possibility of confusion, we use just the name of the vertex to name the angle. Thus in the preceding diagram, $\angle A$ is an acceptable name for $\angle BAC$. Occasionally we will name an angle in a figure by placing a lower-case letter (often from the Greek alphabet) between its sides. For instance, for the angles in the following figure



we can write

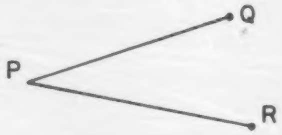
$$\angle MRN = \angle a \quad \text{and} \quad \angle NRP = \angle \alpha$$

(The symbol α is the first letter, "alpha," of the Greek alphabet. You may wish to refer to Page 351 for the complete alphabet.) However, we cannot refer to $\angle R$, since there are three angles which have R as their vertex, and there is no way of telling whether

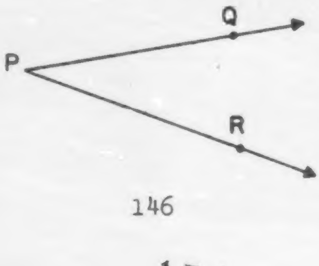
$$\angle MRN \quad \text{or} \quad \angle NRP \quad \text{or} \quad \angle MRP$$

is intended.

We sometimes speak of an angle determined by two non-collinear segments with one common endpoint. If $\overline{PQ}$ and $\overline{PR}$ are the segments,



then the angle which they determine is $\angle QPR$.



Problem Set 4-3

1. Complete this definition of an angle: An angle is the _____ of two _____ which have a common _____ but do not lie in the same _____ .

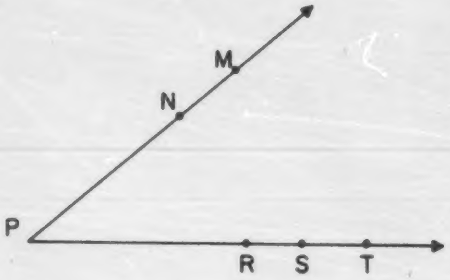
2. (a) Is the union of two opposite rays an angle?
   (b) Is the union of any two rays which do not lie in the same line an angle?
   (c) Justify your answer to each of the above questions by a sketch or by an explanation or both.

3. Complete:

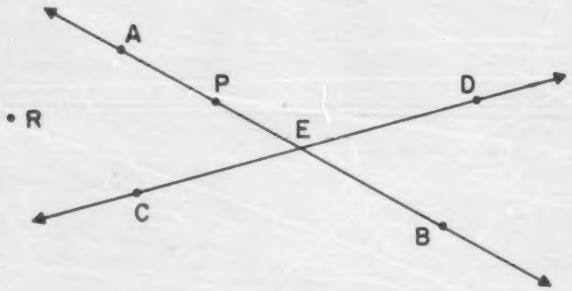
$\angle P = \angle NPA = \angle MPR$

$= \_\_\_\_ = \_\_\_\_$

$= \_\_\_\_ = \_\_\_\_$

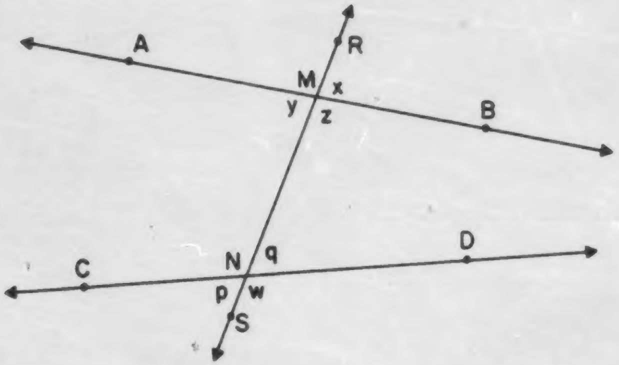


4. Name the angles formed by $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersecting at E .



(a) Does point P belong to $\angle AEC$?
(b) Does point R belong to $\angle AEC$?
(c) Does $\overline{DE}$ belong to $\angle DEB$?
(d) Does the union of $\overline{DE}$ and $\overline{BE}$ belong to $\angle DEB$?
(e) To which angles in the figure does point E belong?

5. Using three letters, write another name for each of the angles designated by a lower case letter in the figure:



(a) ∠x = _____

(b) ∠q = _____

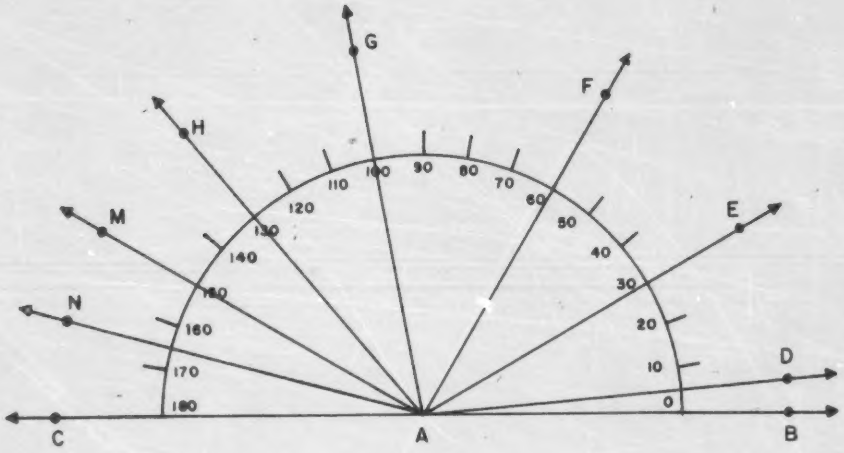
(c) ∠y = _____

(d) ∠w = _____

(e) ∠p = _____

(f) ∠z = _____

Are there any other angles in the figure? What are they?



Using the diagram, compute each of the following. (Be prepared to explain how your answers were obtained.)

(a) m ∠FAB .

(b) m ∠EAB .

(c) m ∠MAC .

(d) m ∠FAE .

(e) m ∠GAE .

(f) m ∠MAN .

(g) m ∠EAD .

(h) m ∠FAG + m ∠GAH

(i) m ∠GAF + m ∠FAE .

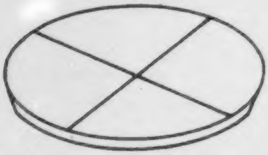
(j) m ∠MAB - m ∠FAB .

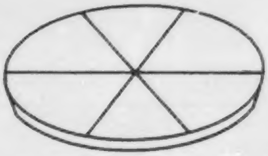
(k) m ∠HAB - m ∠DAB .

(l) m ∠NAE - m ∠NAH .

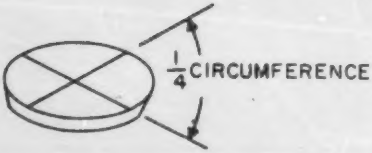
4-4. The Measurement of Angles.

No doubt each of us has at some time been asked to cut a pie, perhaps into four parts of the same size. Now as we stop to think about it, it is clear that when we divide a pie into quarters,

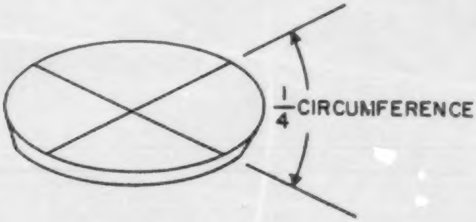




the cuts we make also divide the rim into quarters. Similarly, if we cut a pie into six pieces of the same size, we automatically divide the rim into sixths. Moreover, these observations do not depend upon the size of the pie. Whether the radius of the pie be large or small, cutting it into quarters, for instance, necessarily makes the length of the curved edge of each piece equal to one-quarter of the circumference.



small pie

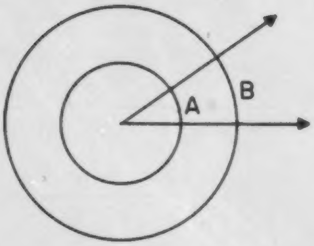


large pie

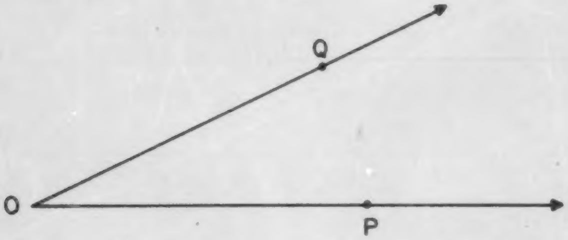
Furthermore, a similar remark applies for any number of pieces into which we wish to cut the pie.

Let us agree, then, that the fractional part of the rim which forms the edge of a piece of pie depends only on the number of pieces we cut and not on the radius of the pie. If we wish our geometry to incorporate these properties of physical objects, we must consider ways of expressing them more abstractly.

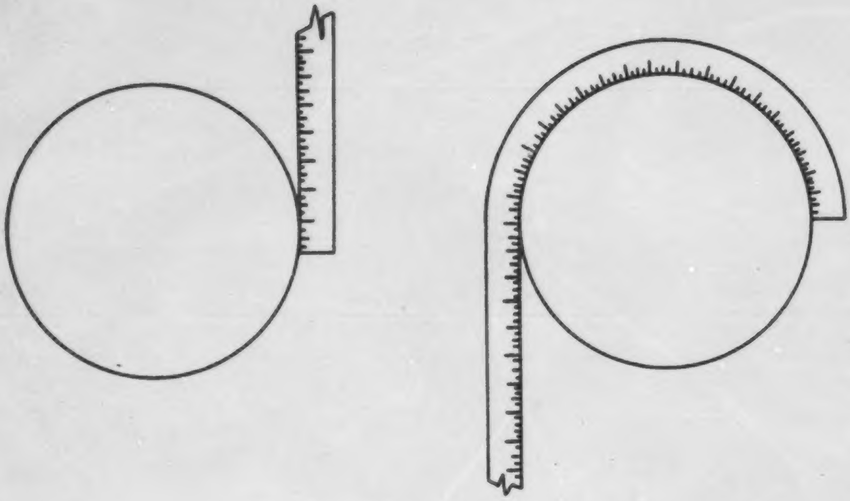
Consider the following figure:



In it we have two circular regions with a common center (these correspond to two pies) and an angle whose vertex is at this center (this cuts out in each circular region a wedge-shaped figure like a piece of pie.) What we have said about pies suggests that the length of arc A is exactly the same fraction of the circumference of the smaller circle that the length of arc B is of the circumference of the larger circle. This in turn suggests that an appropriate way to measure the size of an angle, such as $\angle POQ$ in the following figure,

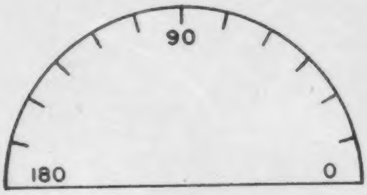


is to draw a circle with center at O and determine what fraction of its circumference is cut off by the sides of the angle. According to our observations about pies, it should make no difference what radius we choose for this circle, since the resulting fraction does not depend on the radius. Equivalently, it appears possible to express the size of an angle simply by giving the length of the circular arc cut off by the angle, provided we know the circumference of the circle we use in the measuring process. We could get this information about a circular object, such as a piece of pie, by wrapping a flexible ruler around it.



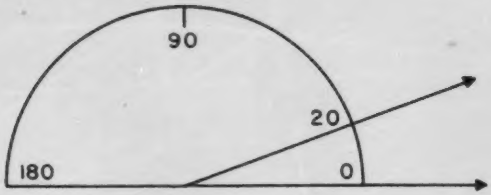
In fact, if we imagine a tape of indefinite length and negligible thickness wrapped around and around the circle, we could measure not only the angles of our geometry but also the amount of rotation of the shaft of an engine or the hands of a clock (provided, of course, that we kept careful track of the rotations and did not read the total length from the wrong loop of the tape!)

The most common device for measuring angles is a protractor.

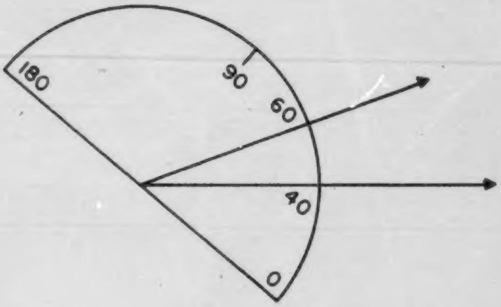


In this country, protractors are usually marked off in degrees. Thus since there are 360 degrees around a full circle, the semicircular edge of a protractor is marked with evenly spaced divisions from 0 to 180 .

Because of the even spacing of the marks on a protractor we can place it like this,

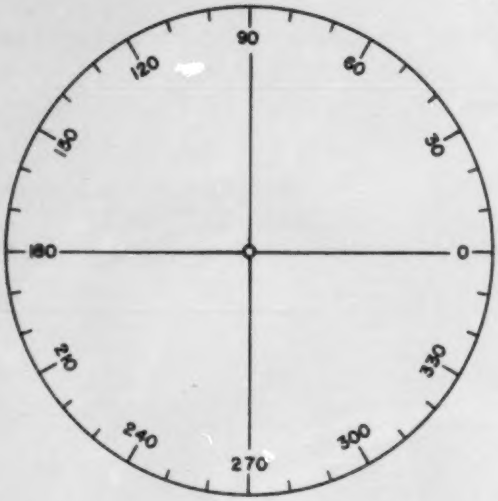


and read the measure of the angle, 20 , directly, or we can place the protractor in a position like this,



and obtain the same answer, 20 , for the measure of the angle by subtracting 40 from 60 .

Another type of protractor is circular rather than semicircular. The 360-degree protractor has advantages in drawing certain figures. It also enables us to measure an angle by subtracting readings on its scale; the method is the same as we illustrated above for the 180-degree protractor.



All our drawings of protractors have shown markings in degrees. From our discussion in Section 3-8 of the role of units in measuring, you might well ask the question: "Are there any other units of angle measure?" Of course the answer is "yes." In fact in some countries where the metric system is used, the scale of a semicircular protractor is divided into 200 equal parts, each of which is called a grad. Engineers who deal with things like rotating shafts of motors and generators commonly use one full revolution as a unit. This means that an engineer's semicircular protractor, at least the one he carries in his mind, would be graduated uniformly from 0 to $\frac{1}{2}$. The army uses another unit, mil, for directing the aim of its artillery. If you continue your study of mathematics, you will meet still another unit, known as the radian, for measuring angles.

If you have ever been on a treasure hunt, you have probably seen directions like "go twelve paces forward, turn right, and take three paces." Most people think in terms of the right angle as a basic unit, and so instructions like this are quite clear. If a protractor were made for this type of measure, how would it be graduated?

There is no particular reason for using degrees in measuring angles, other than the fact that it is commonly done and has the weight of long historical precedent. For convenience we, too, will use degree-measure exclusively. This will, in effect, eliminate from our development postulates similar to Postulates 11 and 13. On the other hand, it does not eliminate logical questions of the kind we raised in Chapter 3 and answered by showing, in Section 3-8, that the fundamental results we needed were true regardless of the unit we used. Similar theorems can be proved for angle measurement also, but we shall omit them in order to proceed more quickly to other topics.

We are now ready to summarize our experiences concerning angle measurement and then formulate our first postulate about angles. We have agreed to measure angles in degrees, and our work with protractors suggests that to every angle there corresponds a number between 0 and 180 .

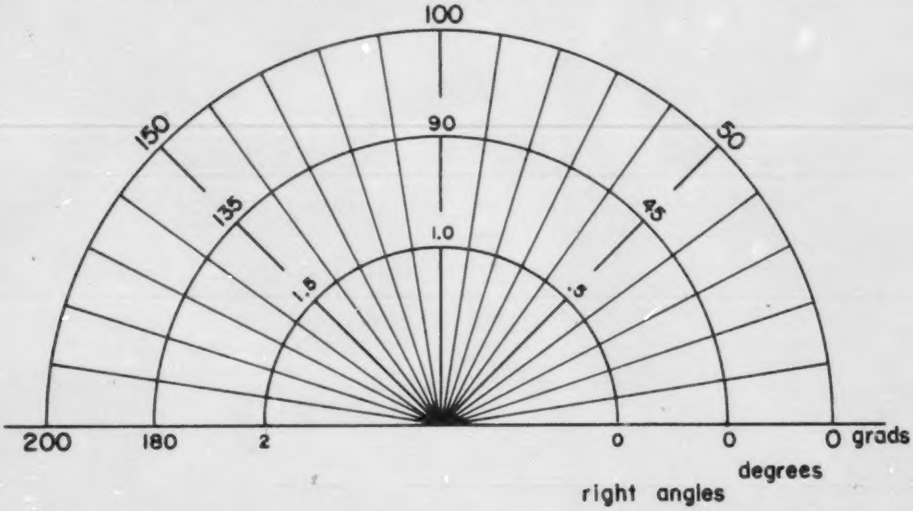
Postulate 16. There exists a correspondence which associates with each angle in space a unique number between 0 and 180 .

DEFINITION. The number which corresponds, by Postulate 16, to an angle is called the measure of the angle.

Notation. If $\angle ABC$ is any angle, its measure is denoted by the symbol $m\angle ABC$.

Problem Set 4-4

1. In the following diagram three different protractors are pictured. As indicated one of these protractors is graduated in right angles, another in degrees, and the third in grads. Using the protractors in this overlay position makes it easier to see three of the different numbers which may be associated with a given angle if different units are used. Thus an angle associated with the number 45 relative to a degree unit is also associated with the number 0.5 relative to a right angle unit and the number 50 relative to a grad unit.



Note that 45 degrees corresponds to $\frac{45}{180}$ or $\frac{1}{4}$ of the semicircle.

.5 right ∠ s corresponds to $\frac{.5}{2}$ of $\frac{1}{4}$ of the semicircle.

50 grads corresponds to $\frac{50}{200}$ or $\frac{1}{4}$ of the semicircle.

(a) x degrees corresponds to what part of a semicircle?

(b) y right angles corresponds to what part of a semicircle?

(c) z grads corresponds to what part of a semicircle?

The phrase "a semicircle" in these statements may be replaced by "2 right angles" or by "180 degrees" or by "200 grads." Thus

18 degrees corresponds to $\frac{18}{180}$ of 200 grads or to 20 grads

.4 right angles corresponds to $\frac{.4}{2}$ or $\frac{1}{5}$ of 180 degrees or to 36 degrees

70 grads corresponds to $\frac{70}{200}$ of 2 right angles or to $\frac{70}{200}$ of 180 degrees.

2. Fill in the blank spaces in the table below so that the three corresponding numbers refer to the same fractional part of the semicircle.

	Degree Unit	Right Angle Unit	Grad Unit
(a)		1.5	
(b)			20
(c)	54		
(d)	x		
(e)		y	
(f)			130
(g)			z

3. Fill in the blank with the appropriate word.

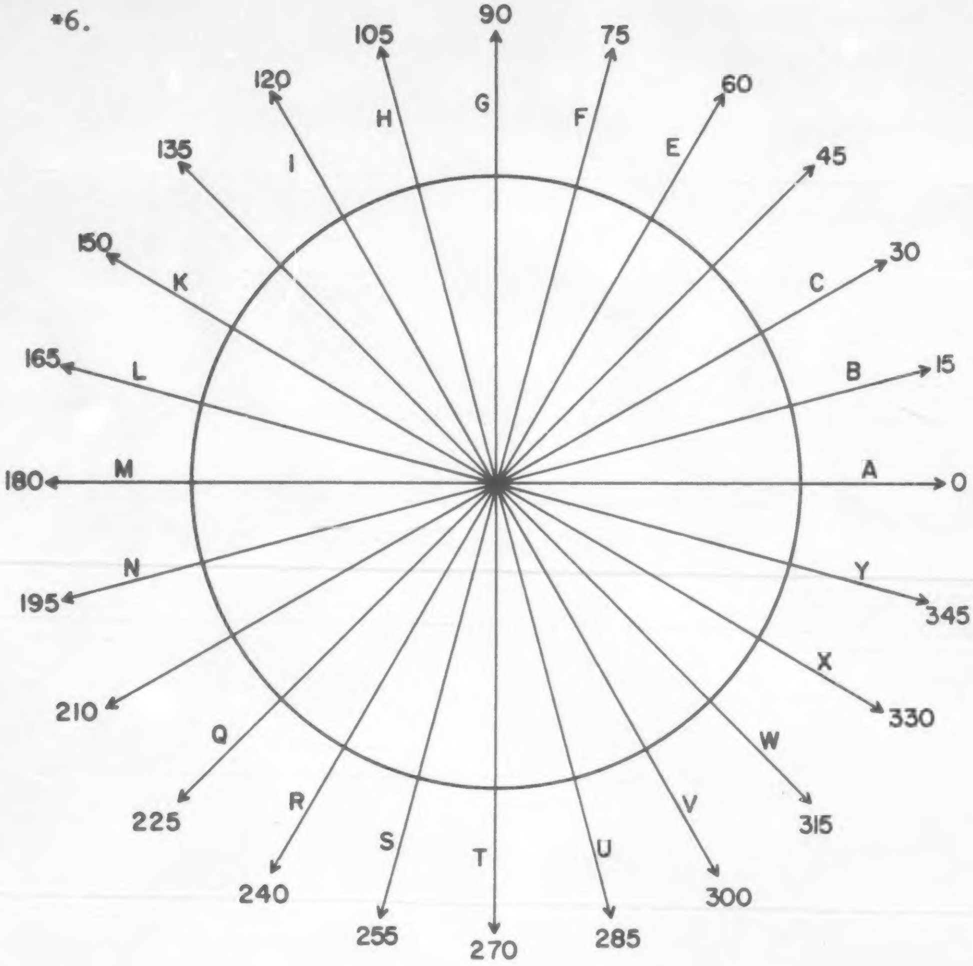
   The smaller the unit used the __________ (larger, smaller) the number associated with a given division on a semicircle.

4. Using the numbers from Parts (a) and (b) of the table which you completed in Problem 2, find the quotient obtained by dividing each number in Part (a) by the number in Part (b) in the same column. How do the three quotients obtained from the different columns compare?

5. (a) Do the same as in Problem 4 using Parts (a) and (c) of Problem 2.

   (b) Do the same as in Problem 4 using Parts (d) and (f) of Problem 2.

*6.



(a) Sometimes it is convenient to use a circular protractor of 360 degrees (like the one pictured above) to find the measure of an angle. From this diagram find the measure of each of the following angles:

(1)	∠AOE	(6)	∠SOC
(2)	∠COH	(7)	∠VOY
(3)	∠FOM	(8)	∠IOX
(4)	∠KOR	(9)	∠QOY
(5)	∠TOA	(10)	∠BOW

(b) Reread the definition of an angle and Postulate 16. Do your answers in (a) agree with this definition and postulate? If not, try to find the measure of the angle which does satisfy our requirements.

(c) Complete the following statements: If p and z are numbers which correspond respectively to $\overrightarrow{OP}$ and $\overrightarrow{OZ}$, and if $p > z$, then the measure of $\angle POZ$ is ______ if this number is less than ______ . If the number is greater than _____ , the measure of the angle is _______ .

4-5. Ray-Coordinate System in a Plane.

In Section 3-5 we described a coordinate system on a line. A coordinate system is a certain one-to-one correspondence between a set of points and a set of numbers which relates, in a specified manner, differences between numbers and distances between points. Later we adopted a postulate (the Ruler Postulate) which provides us, on a given line, with a coordinate system having special properties: a given point is the origin, and every interior point of a given ray with endpoint at the origin has a positive coordinate.

In the same way that the Ruler Postulate gave us a "mathematical ruler," assigning coordinates to points, we want a "mathematical protractor" which will assign "ray-coordinates" to rays. We first introduce the definition describing what we mean by a ray-coordinate system. Afterwards we state the Protractor Postulate which assures us that we have such systems and also tells us that there is just one such system with certain properties. We are guided in our statements by our experience in the real world with the 360-degree protractor which assigns a number to every ray in a plane with a given endpoint.

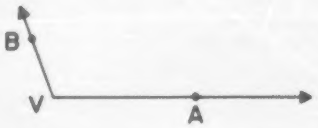
DEFINITION. Let V be a point in a plane $\mathcal{F}$. A ray-coordinate system in $\mathcal{F}$ relative to V is a one-to-one correspondence between the set of all rays in $\mathcal{F}$ with endpoint V and the set of all numbers x such that $0 \leq x < 360$ with the following property: if numbers r and s correspond to rays $\overrightarrow{VR}$ and $\overrightarrow{VS}$ in $\mathcal{F}$ and if $r > s$, then

$$\begin{cases} m\angle RVS = r - s \text{ , if } r - s < 180 \text{ ;} \\ m\angle RVS = 360 - (r - s) \text{ , if } r - s > 180 \text{ ;} \\ \overrightarrow{VR} \text{ and } \overrightarrow{VS} \text{ are opposite rays, if and only if} \\ \qquad r - s = 180 \text{ .} \end{cases}$$

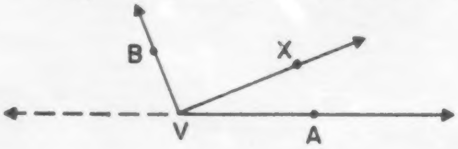
DEFINITIONS. The number which a given ray-coordinate system assigns to a ray is called the ray-coordinate of the ray. The ray whose ray-coordinate is zero is called the zero-ray of the ray-coordinate system.

Postulate 17. (The Protractor Postulate) If $\mathcal{F}$ is any plane and if $\overrightarrow{VA}$ and $\overrightarrow{VB}$ are noncollinear rays in $\mathcal{F}$, then there is a unique ray-coordinate system in $\mathcal{F}$ relative to V such that $\overrightarrow{VA}$ corresponds to 0 and such that every ray $\overrightarrow{VX}$ with X and B on the same side of $\overleftrightarrow{VA}$ corresponds to a number less than 180.

Let us interpret the Protractor Postulate in a diagram. Suppose that A, B, V are three noncollinear points. We know they determine a plane, which we may call $\mathcal{F}$. They also determine two noncollinear rays with endpoint V, namely $\overrightarrow{VA}$ and $\overrightarrow{VB}$.



Consider the halfplane $\mathcal{H}$ which is the side of $\overleftrightarrow{VA}$ containing B . The Postulate assures us that there is one, and only one, ray-coordinate system such that $\overrightarrow{VA}$ is the zero-ray, and for any point X in $\mathcal{H}$, the ray-coordinate of $\overrightarrow{VX}$ is less than 180 .

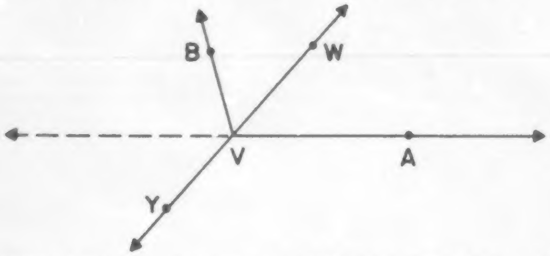


In particular, $\overrightarrow{VB}$ has ray-coordinate less than 180 .

An immediate consequence of Postulate 17 is the following useful theorem.

THEOREM 4-3. (Angle Construction Theorem) If $\mathcal{H}$ is a halfplane whose edge contains the ray $\overrightarrow{VA}$ and if r is any number between 0 and 180 , then there is a unique ray $\overrightarrow{VR}$ such that R is in $\mathcal{H}$ and $m\angle AVR = r$.

As another application of the Protractor Postulate, suppose that Y is a point in the plane $\mathcal{F}$ such that Y and B are on opposite sides of $\overleftrightarrow{VA}$. Now $\overrightarrow{VW}$, the ray opposite to $\overrightarrow{VY}$, has a ray-coordinate less than 180 , say w .



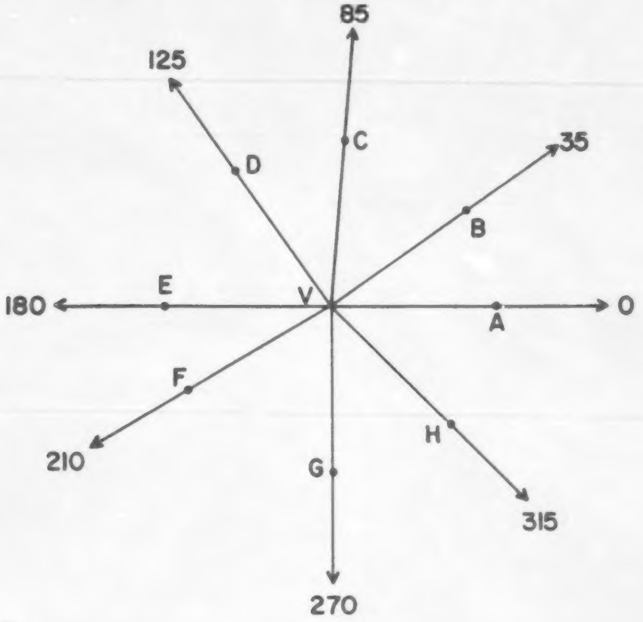
By the definition of a ray-coordinate system, the ray-coordinate y of $\overrightarrow{VY}$ satisfies the condition that $y - w = 180$. Hence y is greater than 180 .

In summary, the ray-coordinate of a ray is less than, or greater than, 180 according as point B and the interior of the ray lie on the same side, or on opposite sides, of $\overleftrightarrow{VA}$.

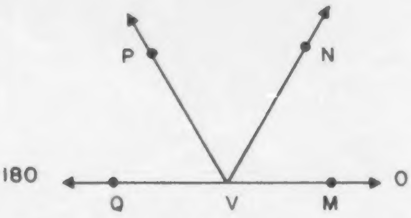
Problem Set 4-5

1. Given that a one-to-one correspondence, as described in the definition of a ray-coordinate system, assigns to the rays $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$, $\overrightarrow{VD}$, $\overrightarrow{VE}$, $\overrightarrow{VF}$, $\overrightarrow{VG}$ and $\overrightarrow{VH}$ the real numbers indicated in the accompanying diagram. Find the measure of each of the following angles:

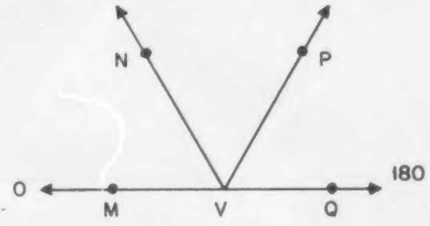
   (a) $\angle AVB$
   (b) $\angle AVC$
   (c) $\angle BVC$
   (d) $\angle AVH$
   (e) $\angle DVG$
   (f) $\angle CVE$
   (g) $\angle CVD$
   (h) $\angle DVB$
   (i) $\angle BVG$
   (j) $\angle CVH$



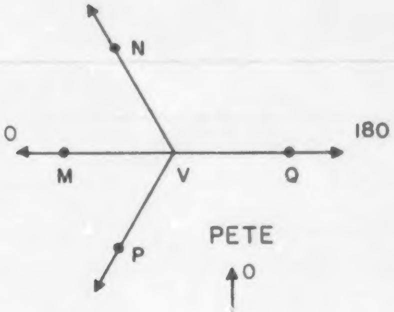
2. The students in a class were given a ray-coordinate system (as described in the Protractor Postulate) which assigned to the rays $\overrightarrow{VM}$, $\overrightarrow{VN}$, $\overrightarrow{VP}$, $\overrightarrow{VQ}$ the real numbers 0 , 60 , 130 and 180 , respectively. Five boys, Tom, Jim, Bill, Hank, and Pete, were each asked to illustrate the problem. They submitted the following drawings:



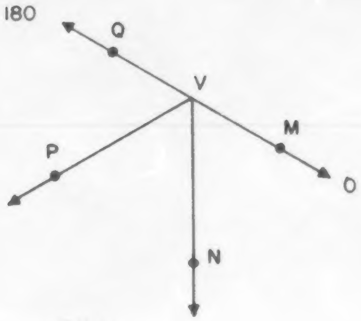
TOM



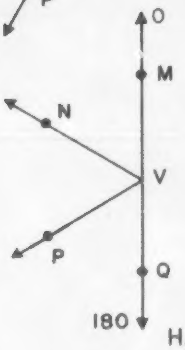
JIM



PETE



BILL



HANK

(a) If you were the boys' teacher, which of these illustrations would you accept as correct? Why?

(b) Find the measures of $\angle MVN$ and $\angle MVP$. Justify your answer.

(c) Find the measure of $\angle NVP$. Justify your answer.

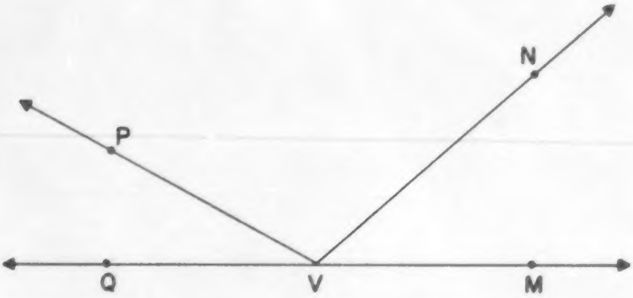
3. Let $\overrightarrow{VM}$, $\overrightarrow{VA}$, $\overrightarrow{VC}$, $\overrightarrow{VE}$ be coplanar rays such that no two of them are collinear. Let a ray-coordinate system have zero-ray $\overrightarrow{VM}$ and assign to the rays $\overrightarrow{VA}$, $\overrightarrow{VC}$, $\overrightarrow{VE}$ the real numbers x, y, z, respectively, where $x < y < z$.

   (a) Suppose that $z < 180$. Find the measure of each of the following angles:

   (1) $\angle AVC$ (3) $\angle CVE$
   (2) $\angle AVE$ (4) $\angle MVC$

   (b) Find the measures of the angles in Part (a) in case $z > 180$.

4.



Referring to the above figure, describe:

(a) the union of $\angle MVN$ and $\angle NVP$; the union of $\angle MVN$ and $\angle QVP$;

(b) the intersection of the following pairs of angles

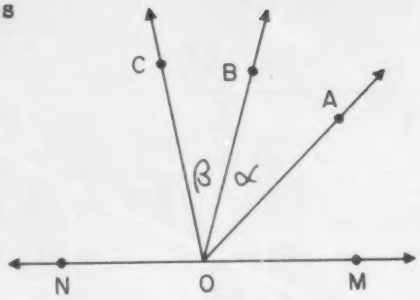
(1) $\angle MVN$, $\angle NVP$ (3) $\angle NVM$, $\angle PVQ$
(2) $\angle MVP$, $\angle NVQ$ (4) $\angle NVQ$, $\angle QVP$

(c) Given that $\overleftrightarrow{QM}$ is the edge of the halfplane $\mathcal{H}$, that $\mathcal{H}$ contains N and P, and that V is a point on $\overleftrightarrow{QM}$, describe:

(1) the intersection of $\mathcal{H}$ and $\overleftrightarrow{QM}$;
(2) the union of $\mathcal{H}$ and $\overleftrightarrow{QM}$;
(3) the intersection of $\angle PVN$ and $\mathcal{H}$.

5. Suppose that for the angles shown in the figure at the right,

$m\,\angle MOA = 40$
$m\,\angle AOB = 30$
$m\,\angle BOC = 30$
$m\,\angle CON = 80$.



Are the following statements true or false in our development of geometry? If your answer is false, explain.

(a) $\angle AOB = \angle BOC$.
(b) $\angle\alpha = \angle BOA$.
(c) $\angle COB = \angle BOC$.
(d) $m\,\angle AOM < m\,\angle\beta$.
(e) $m\,\angle AOM = \frac{1}{2}\,m\,\angle CON$.
(f) $\angle\alpha = \angle\beta$.
(g) $m\,\angle COA = 2m\,\angle\alpha$.
(h) $m\,\angle MOB + m\,\angle AOC + m\,\angle CON = 210$

6. Which of the following expressions are meaningless in our development of geometry? Explain your answers.

(a) $\angle RVY = 80$.
(b) $m\,\angle YVR = 2m\,\angle XVP$.
(c) $\angle PVX + \angle PVQ = \angle XVQ$.
(d) $\angle a = \frac{1}{2}\,m\,\angle PVR$.
(e) $\angle\alpha + \angle\beta = 90$.
(f) $\angle QVP = \angle PVQ = \angle\varphi$.
(g) $m\,\angle c + m\,\angle d > \angle PVX$.
(h) $\angle PVR = 2\,\angle PVQ$.
(i) $m\,\angle YVX = 180$.
(j) $m\,\angle YVR = 280$.

7. (a) Given $\overrightarrow{AC}$ lying in the edge of a halfplane $\mathcal{H}$ and a number r between 0 and 180 , how many rays $\overrightarrow{AB}$ extend into $\mathcal{H}$ such that $m\,\angle BAC = r$? Why?
(b) Given $\overrightarrow{AC}$ lying in a plane $\mathcal{E}$, and a number r between 0 and 180 , how many rays $\overrightarrow{AB}$ are there in $\mathcal{E}$ such that $m\,\angle BAC = r$? Why?

4-6. Betweenness for Rays.

We have not yet discussed what we mean by saying that one ray is between two others, but nonetheless the idea probably has some meaning for you. Using your present notions of betweenness, decide in which of the figures below, exactly one of the rays is between the other two.

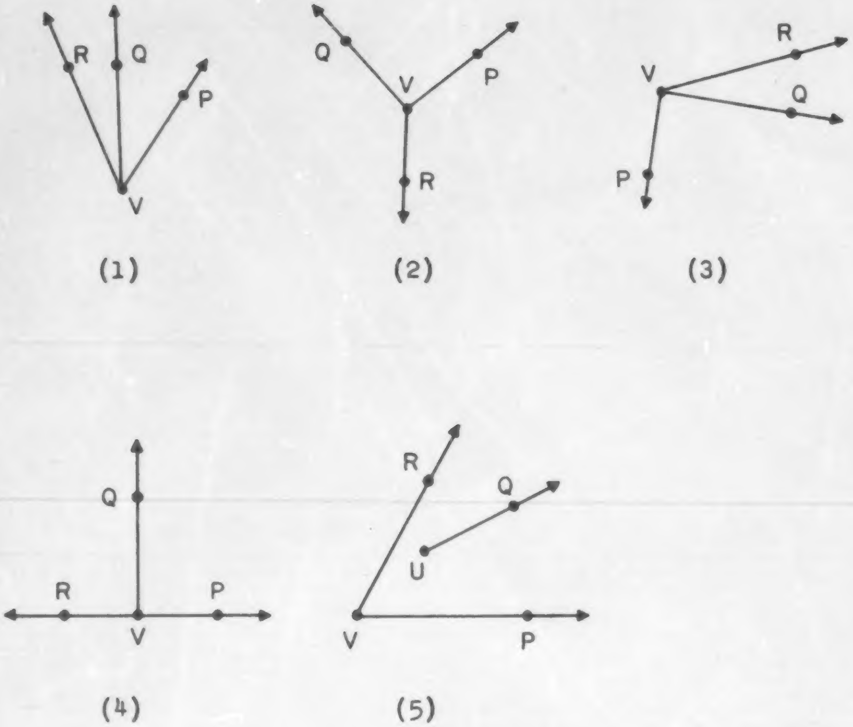
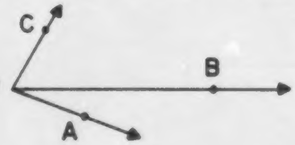


Figure (a)

Our description of betweenness for three collinear points relied heavily on the use of a coordinate system on the line containing the points. We choose to describe betweenness for three rays in terms of a ray-coordinate system. This decision means that the three rays must be coplanar and concurrent.

> DEFINITION. If three concurrent rays, $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$ in a plane $\mathcal{F}$ are given, then $\overrightarrow{VB}$ is said to be between $\overrightarrow{VA}$ and $\overrightarrow{VC}$ if and only if there is a ray-coordinate system in $\mathcal{F}$ relative to V such that the respective ray-coordinates 0 , b , c of $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$ satisfy the condition that $0 < b < c < 180$.

Suppose $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$. Then the ray-coordinates mentioned in the definition satisfy $0 < b < c < 180$. Thus the points B and C lie on the same side of $\overleftrightarrow{VA}$.



From the definition of a ray-coordinate system, $m\angle CVA = c$ and $m\angle CVB = c - b$.

Another ray-coordinate system in the plane has VC as zero-ray and assigns new ray-coordinates as shown in the table:

	$\overrightarrow{VA}$	$\overrightarrow{VB}$	$\overrightarrow{VC}$	$m\angle CVA$	$m\angle CVB$	$m\angle BVA$
Old system	0	b	c	c	c - b	b
New system	c	c - b	0	c	c - b	b

Since each of the numbers c and c - b is less then 180, we see that the points B and A lie on the same side of $\overleftrightarrow{VC}$.

In Figure (a) at the beginning of this section, the indicated ray containing Q appears to be between $\overrightarrow{VP}$ and $\overrightarrow{VR}$ in Parts (1) and (3), but not in the others. In Part (2), Q and R are not on the same side of $\overleftrightarrow{VP}$. In Part (4), two of the rays appear to be collinear. Why does our definition say that $\overrightarrow{UQ}$ is not between $\overrightarrow{VP}$ and $\overrightarrow{VR}$ in Part (5)?

The proof of the next theorem resembles the proof of Theorem 3-9, and we shall leave it as a problem.

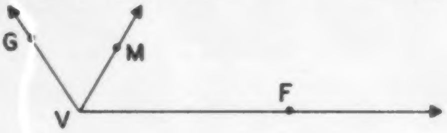
THEOREM 4-4. (The Betweenness-Angles Theorem) Let $\overrightarrow{VE}$, $\overrightarrow{VF}$, $\overrightarrow{VG}$ be rays such that $\overrightarrow{VF}$ is between $\overrightarrow{VE}$ and $\overrightarrow{VG}$. Then $m\angle EVF + m\angle FVG = m\angle EVG$ (or, $m\angle EFV = m\angle EVG - m\angle FVG$.)

In our discussion of betweenness for points, we introduced the phrase "collinear in that order." A useful terminology for rays is "concurrent in that order." We shall sometimes speak of three rays $\overrightarrow{VP}$, $\overrightarrow{VQ}$, $\overrightarrow{VR}$ as being concurrent in that order to mean that they are coplanar and that $\overrightarrow{VQ}$ is between $\overrightarrow{VP}$ and $\overrightarrow{VR}$. Occasionally we shall refer to four rays $\overrightarrow{VP}$, $\overrightarrow{VQ}$, $\overrightarrow{VR}$, $\overrightarrow{VS}$ as being concurrent in that order. By this we shall mean that the four rays are coplanar, that $\overrightarrow{VQ}$ is between $\overrightarrow{VP}$ and $\overrightarrow{VR}$, that $\overrightarrow{VR}$ is between $\overrightarrow{VQ}$ and $\overrightarrow{VS}$, and also that each of $\overrightarrow{VQ}$ and $\overrightarrow{VR}$ is between $\overrightarrow{VP}$ and $\overrightarrow{VS}$. Thus, if $\overrightarrow{VP}$, $\overrightarrow{VQ}$, $\overrightarrow{VR}$, $\overrightarrow{VS}$ are concurrent in that order, there is a ray-coordinate system in which the respective ray-coordinates 0, q, r, s of the four rays satisfy $0 < q < r < s < 180$.

The midpoint of a segment is a special case of a point between two points. An important particular case of a ray between two rays is the "midray" of the angle which they form.

DEFINITION. A ray is called the midray of an angle if the ray is between the sides of the angle and forms with them two angles of equal measure.

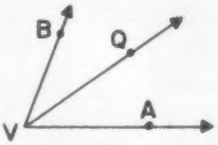
In symbols, the ray $\overrightarrow{VM}$ is the midray of $\angle FVG$ if and only if $\overrightarrow{VM}$ is between $\overrightarrow{VF}$ and $\overrightarrow{VG}$ and $m\angle FVM = m\angle MVG$.



The proof of the next theorem resembles the proof of Theorem 3-3, and we shall leave it as a problem.

THEOREM 4-5. Every angle has a unique midray.

DEFINITION. The midray of an angle is said to bisect the angle and is called the angle bisector.



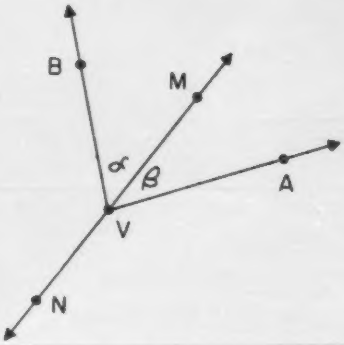
Thus the angle bisector of $\angle AVB$ is the ray $\overrightarrow{VQ}$ between $\overrightarrow{VA}$ and $\overrightarrow{VB}$ such that $m\angle AVQ = \frac{1}{2} \cdot m\angle AVB = m\angle BVQ$.

In a ray-coordinate system, suppose that the ray-coordinates of $\overrightarrow{VA}$ and $\overrightarrow{VB}$ are a and b, respectively. If $a < b$ and $b - a < 180$, then the ray-coordinate of the midray is $\frac{a + b}{2}$.

Problem Set 4-6

1. If a ray-coordinate system in a plane assigns to the rays $\overrightarrow{VN}$, $\overrightarrow{VP}$, $\overrightarrow{VM}$ the ray-coordinates 60 , 25 , 108 respectively, which of the three rays is between the other two? What is $m\angle NVP$?

2. In the accompanying figure, if $m\angle\alpha = m\angle\beta$, then $\overrightarrow{VM}$ is called the _______ of $\angle AVB$ and is said to _______ this angle.

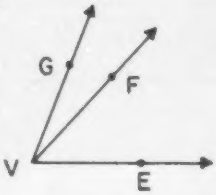


3. Let x, y, z be the real numbers which a one-to-one correspondence described by the Protractor Postulate assigns to rays $\overrightarrow{VP}$, $\overrightarrow{VN}$, $\overrightarrow{VM}$, respectively. Draw a figure illustrating the relative positions of the rays for each of the following cases:

   (a) $x < y < z < 180$.
   (b) $z < x < y < 180$.
   (c) $y < z < x < 180$.

*4. Reread Theorem 3-9 and its proof before completing the following proof of Theorem 4-4.

   By hypothesis, $\overrightarrow{VF}$ is between $\overrightarrow{VE}$ and $\overrightarrow{VG}$. By the definition of ________ for rays, there is a ray-coordinate system such that the respective ray-coordinates 0 , f , g of $\overrightarrow{VE}$, $\overrightarrow{VF}$, $\overrightarrow{VG}$ satisfy $0 < f < g < 180$. By the definition of a ____________ system,



$$m\angle EVF = f - 0 = f\ ,$$

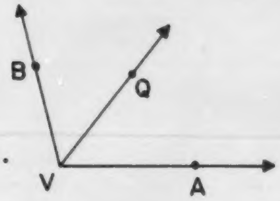
$$m\angle FVG = \underline{\qquad\qquad}\ ,$$

and $$m\angle EVG = \underline{\qquad\qquad}\ .$$

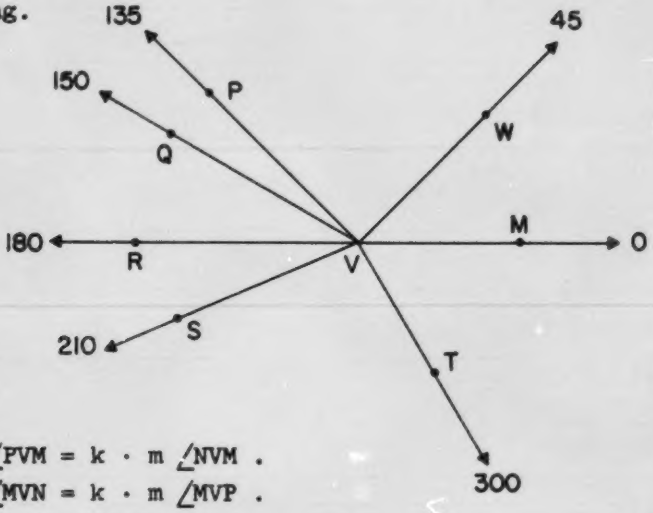
Finish the task of showing that $m\angle EVF + m\angle FVG = m\angle EVG$. Then explain why $m\angle EVF = m\angle EVG - m\angle FVG$.

*5. Reread Theorem 3-3 and its proof before completing the following proof of Theorem 4-5.

Let $\angle AVB$ be any angle. There is a ray-coordinate system such that $\overrightarrow{VA}$ is the zero-ray and the ray-coordinate of $\overrightarrow{VB}$ is a number, say b , less than _____ . If q is the ray-coordinate of the desired midray $\overrightarrow{VQ}$, then the requirement that the midray is _______ the sides of $\angle AVB$ may be expressed by $0 < q <$ _____ . The second requirement for the desired midray is that $m\angle AVQ = m\angle$ ____ . The _______ of a ray-coordinate system tells us that $m\angle AVQ =$ _____ and that $m\angle$ ____ $= b - q$. Thus the second requirement becomes the equation $q = b - q$. Explain why this equation has exactly one solution, namely $q = \frac{b}{2}$. Explain why there is exactly one ray whose ray-coordinate in the chosen ray-coordinate system is $\frac{b}{2}$. Is the ray whose ray-coordinate is $\frac{b}{2}$ the desired midray of $\angle AVB$? Why? Does $\angle AVB$ have only one midray? Why?



6. Given a ray-coordinate system in which ray-coordinates 0 , 45 , 135 , 150 , 180 , 210 , 300 are assigned to $\overrightarrow{VM}$, $\overrightarrow{VN}$, $\overrightarrow{VP}$, $\overrightarrow{VQ}$, $\overrightarrow{VR}$, $\overrightarrow{VS}$, $\overrightarrow{VT}$, respectively. Find the real number k in each of the following.



(a) $m\,\angle PVM = k \cdot m\,\angle NVM$.
(b) $m\,\angle MVN = k \cdot m\,\angle MVP$.
(c) $m\,\angle NVM = k \cdot m\,\angle NVQ$.
(d) $m\,\angle RVP = k \cdot m\,\angle NVP$.
(e) $m\,\angle TVM = k \cdot m\,\angle QVR$.
(f) $m\,\angle SVT = k \cdot m\,\angle NVP$.
(g) $m\,\angle SVT = k \cdot m\,\angle QVS$.
(h) $m\,\angle NVT = k \cdot m\,\angle MVS$.

7. Given the halfplane $\mathcal{H}$ whose edge contains the point V , and three concurrent rays, $\overrightarrow{VP}$, $\overrightarrow{VN}$, $\overrightarrow{VM}$, with points M , N , P in $\mathcal{H}$. If $m\,\angle NVP = 48$, $m\,\angle PVM = 83$, and $m\,\angle MVN = 35$, which (if any) of the three rays is between the other two?

8. If a one-to-one correspondence, as described by the Protractor Postulate, has assigned to the rays $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, the real numbers 32, 80, and q respectively, find q when
   (a) $\overrightarrow{AD}$ is the midray of $\angle BAC$.
   (b) $\overrightarrow{AC}$ is the midray of $\angle BAD$.
   (c) $\overrightarrow{AB}$ is the midray of $\angle CAD$.

9. Find the ray-coordinate of the midray of $\angle XOY$ in terms of the ray-coordinates x and y of $\overrightarrow{OX}$ and $\overrightarrow{OY}$, respectively, in each of the following cases:
   (a) $x > y$ and $x - y < 180$.
   (b) $x > y$ and $x - y > 180$.

10. Suppose that a ray-coordinate system assigns ray-coordinates $0, n, p, q$ to $\overrightarrow{VM}$, $\overrightarrow{VN}$, $\overrightarrow{VP}$, $\overrightarrow{VQ}$, respectively, where $0 < n < p < q < 180$.
    (a) Find k in terms of n and p if $m\angle MVN = k \cdot m\angle PVN$.
    (b) Find k in terms of n, p, q if $m\angle NVP = k \cdot m\angle NVQ$.
    (c) Find p in terms of n and q if $m\angle NVP = \frac{2}{3} \cdot m\angle NVQ$.

11. Suppose that a ray-coordinate system assigns the numbers d and e to the rays $\overrightarrow{AD}$ and $\overrightarrow{AE}$, respectively, where $d < e < 180$. If $\overrightarrow{AP}$ is a ray between $\overrightarrow{AD}$ and $\overrightarrow{AE}$, such that

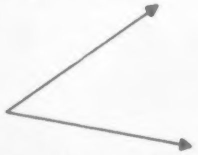
$$\frac{m\angle DAP}{m\angle DAE} = h, \quad (0 < h < 1)$$

show that the number assigned to the ray $\overrightarrow{AP}$ by the correspondence is

$$p = (1 - h)d + he.$$

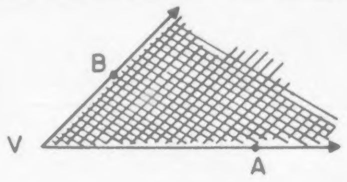
4-7. Interior of an Angle.

We have already discussed how a line in a plane separates the plane into two halfplanes. A picture of an angle clearly indicates that the angle separates the plane into two parts,

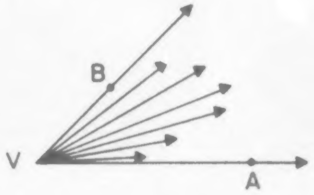


one of which appears to be "inside" and the other "outside" the angle. There are several ways in which we may think of the inside, or, as we prefer to say, the interior of an angle. Let us consider some of these notions before we give the definition of the interior of an angle.

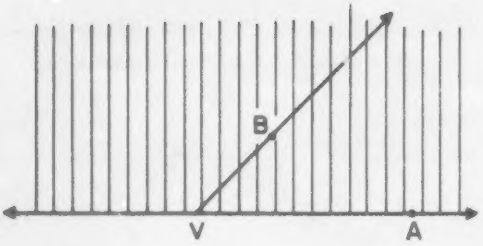
In the diagram the interior of the angle $\angle AVB$ is cross-hatched.

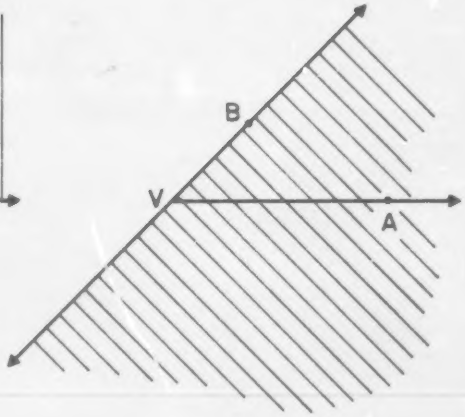


One approach is to notice that the cross-hatched region consists of the interior points of all rays which are between the sides of the angle.

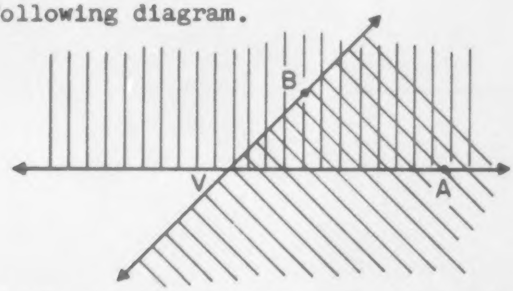


Another approach is to notice that the cross-hatched region consists of all points which are in two halfplanes, namely the side of $\overleftrightarrow{VA}$ which contains B and the side of $\overleftrightarrow{VB}$ which contains A . These two halfplanes are indicated in separate pictures below.



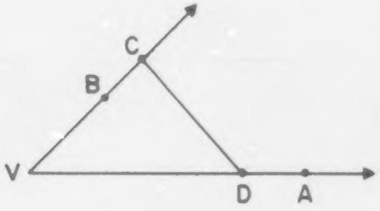


The intersection of the two halfplanes is the cross-hatched region in the following diagram.

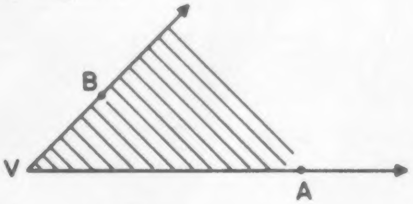


This intersection appears to be what we mean by the interior of the angle.

A third approach is suggested by considering a segment such as $\overline{CD}$ whose two endpoints lie on different sides of the angle.



Every interior point of $\overline{CD}$ appears to be inside the angle. Furthermore the interior of the angle seems to consist of the interior points of all the segments which join points on different sides of the angle.



Let us summarize our three ways of describing the region which we have thought of as the "interior of the angle $\angle AVB$." In precise language, they are: (1) the union of the interiors of all rays between $\overrightarrow{VA}$ and $\overrightarrow{VB}$; (2) the intersection of the halfplane with edge $\overleftrightarrow{VA}$ and containing B and the halfplane with edge $\overleftrightarrow{VB}$ and containing A; (3) the union of the interiors of all segments joining an interior point of $\overrightarrow{VA}$ and an interior point of $\overrightarrow{VB}$. Each of these three descriptions identifies a certain set in the plane AVB. Thus we apparently have three sets to consider. Our experiences may suggest that these sets are the same. It is, in fact, possible to prove that the first and second of these sets are the same, and also to prove that the third of these sets is contained in each of the others. However we can not, at this stage in our development, prove that the third of these sets is the same as the others. Instead of pausing to prove as theorems these results which we need, we prefer to state them as our next postulate.

Postulate 18. (The Interior of an Angle Postulate)

If $\angle AVB$ is any angle,

(1) let $\mathcal{R}$ be the set of all interior points of rays between $\overrightarrow{VA}$ and $\overrightarrow{VB}$,

(2) let $\mathcal{H}$ be the set of all points which belong both to the halfplane with edge $\overleftrightarrow{VA}$ and containing B and to the halfplane with edge $\overleftrightarrow{VB}$ and containing A, and

(3) let $\mathcal{S}$ be the set of all interior points of segments joining an interior point of $\overrightarrow{VA}$ and an interior point of $\overrightarrow{VB}$.

Then $\mathcal{R}$ and $\mathcal{H}$ are the same set, and this set contains $\mathcal{S}$.

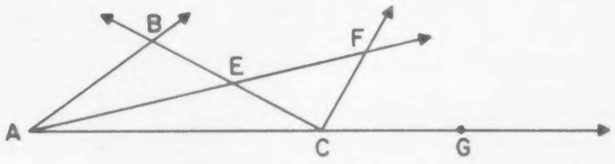
DEFINITION. If $\angle AVB$ is any angle, then the subset of the plane AVB, which was denoted by either $\mathcal{R}$ or $\mathcal{L}$ in Postulate 18, is called the *interior of the angle* $\angle AVB$; and any point belonging to this set is called an *interior point of* $\angle AVB$.

According to this definition, the interior of an angle, in our formal geometry, may be thought of in either one of the two ways we prefer. It may be thought of as a union of interiors of certain rays, or as an intersection of two particular halfplanes. As a simple example, we prove the following theorem.

THEOREM 4-6. The interior of any angle is a convex set.

Proof: According to Postulate 18, the interior of an angle is the intersection of two halfplanes. According to the Plane Separation Postulate, each of these halfplanes is a convex set. Our conclusion then follows from Theorem 4-1, which asserts that an intersection of two convex sets is a convex set.

As a second illustration, we prove the next theorem, which will have an important bearing on our development in Chapter 5. The following picture will improve our understanding of the statements in the theorem and its proof. This does not mean, however, that the appearance of the picture can be a basis for any of the reasons given in steps of the proof.



THEOREM 4-7. Let A, B, C, E, F, G be coplanar points such that A, B, C are not collinear, E is between B and C, the rays $\overrightarrow{EF}$ and $\overrightarrow{EA}$ are opposite, and the rays $\overrightarrow{CG}$ and $\overrightarrow{CA}$ are opposite. Then $\overrightarrow{CF}$ is between $\overrightarrow{CB}$ and $\overrightarrow{CG}$.

Proof:

(1) E is between C and B, by hypothesis.
(2) $\overrightarrow{AE}$ is between $\overrightarrow{AC}$ and $\overrightarrow{AB}$, by Postulate 18.
(3) $\overrightarrow{AF}$ is between $\overrightarrow{AG}$ and $\overrightarrow{AB}$, because $\overrightarrow{AF} = \overrightarrow{AE}$.
(4) F is on the same side of $\overleftrightarrow{CG}$ as B, by Postulate 18.
(5) F and A are on opposite sides of $\overleftrightarrow{BC}$, because $\overleftrightarrow{BC}$ contains point E between F and A.
(6) G and A are on opposite sides of $\overleftrightarrow{BC}$, because $\overleftrightarrow{BC}$ contains point C between G and A.
(7) F is on the same side of $\overleftrightarrow{BC}$ as G, from Statements (5) and (6).
(8) F is in the interior of $\angle GCB$, from Statements (4) and (7) and by Postulate 18.
(9) $\overrightarrow{CF}$ is between $\overrightarrow{CB}$ and $\overrightarrow{CG}$, by Postulate 18.

Now that we have discussed the interior of an angle it is appropriate to consider what we mean by the exterior of an angle. Try to make up a definition which expresses what you mean by the exterior of an angle, before you read the next definition.

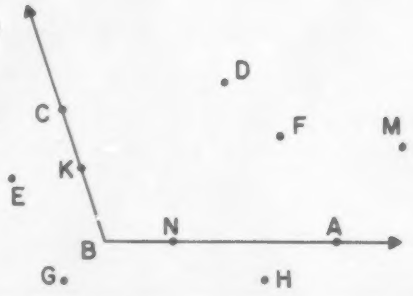
DEFINITION. The exterior of an angle is the set of all points in the plane of the angle which do not lie on the angle and do not lie in the interior of the angle.

Is the exterior of an angle a convex set? Is it the intersection of two halfplanes? Is it the intersection of any finite number of halfplanes? Is it the union of two halfplanes?

Problem Set 4-7

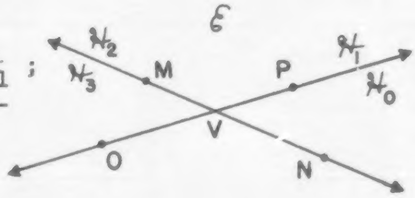
1. (a) Name the points of the figure which are in the interior of $\angle CBA$.

   (b) Name the points of the figure in the exterior of $\angle B$.



2. Is the vertex of an angle in the interior of the angle? In the exterior? Explain.

3. In the figure on the right, $\overleftrightarrow{OP}$ separates plane $\mathcal{E}$ into halfplanes $\mathcal{H}_0$ and $\mathcal{H}_1$; $\overleftrightarrow{MN}$ separates $\mathcal{E}$ into halfplanes $\mathcal{H}_2$ and $\mathcal{H}_3$, as indicated.



   (a) Copy the diagram and shade the intersection of $\mathcal{H}_1$ and $\mathcal{H}_3$.

   (b) Is this shaded portion the interior of any of the angles shown in the figure? The exterior?

   (c) Copy the diagram and shade the union of $\mathcal{H}_1$ and $\mathcal{H}_3$. Is this shaded portion the interior of any of the angles shown in the figure? The exterior?

4. Draw $\angle ABC$. Choose points X and Y in the interior of $\angle ABC$ and P and Q in the exterior.

   (a) Must every point of $\overline{XY}$ be in the interior of $\angle ABC$? Why?

   (b) For your choice of P and Q , is every point of $\overline{PQ}$ in the exterior of $\angle ABC$? Is this true for every choice of points P and Q in the exterior?

   (c) Are there points R and S in the exterior of $\angle ABC$ such that the intersection of $\overline{RS}$ and $\angle ABC$ is not empty?

   (d) Can the intersection of $\overline{XP}$ and $\angle ABC$ be empty?

5. If $m\angle AOB = 100$ and $m\angle BOC = 30$, what is $m\angle AOC$ if

   (a) C is in the interior of $\angle AOB$?

   (b) C is in the exterior of $\angle AOB$?

6. If Y is in the interior of $\angle XOZ$, complete the following:

   (a) $m\angle XOY + m\angle YOZ = m\angle$ ___ .

   (b) $m\angle XOZ - m\angle XOY = m\angle$ ___ .

   (c) $m\angle XOZ - m\angle$ ___ $= m\angle XOY$.

7. Suppose that $m\angle AOB = 70$ and $m\angle BOC = 35$, and that the two angles are coplanar.

   (a) Is $\overrightarrow{OB}$ between $\overrightarrow{OA}$ and $\overrightarrow{OC}$?

   (b) If $m\angle AOC = 105$, is $\overrightarrow{OB}$ between $\overrightarrow{OA}$ and $\overrightarrow{OC}$?

   (c) If $m\angle AOC = 35$, which of the rays $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ is between the other two? What special name might be used in describing this ray?

8. (a) Is the exterior of an angle a convex set? Make a sketch to explain your answer.

   (b) Is the exterior of an angle:

      (1) the intersection of two halfplanes?

      (2) the union of two halfplanes?

   (c) Make a sketch to illustrate your answer to Part (b).

9. Let $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{AE}$ be four rays, concurrent in that order. If $m\angle BAC = m\angle DAE$, then $m\angle BAD = m\angle$ ___ . (Explain your answer.)

10. Let $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{AE}$ be four rays, concurrent in that order. If $m\angle CAE = m\angle DAB$, explain why $m\angle BAC = m\angle DAE$.

11. Show that the union of an angle and its interior is a convex set. Hint: Let X and Y be two distinct points in the union of $\angle AOB$ and its interior. Consider the following five cases.

    (a) X and Y are in the interior of $\angle AOB$.

    (b) X and Y are in the same ray of $\angle AOB$.

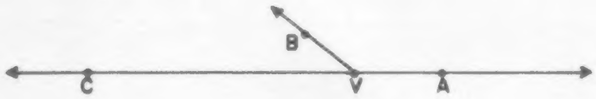
    (c) X and Y are points other than the vertex and on different rays of the angle.

(d) One of the points is in the interior of the angle and the other point is the vertex of the angle.

(e) One of the points is in the interior of the angle and the other point is on one of the rays of the angle but not the vertex 0 .

4-8. Right Angles and Perpendicularity.

Although, for the reasons outlined in Section 4-3, we have chosen to exclude "straight angles" from our definition of angles, we often need to consider figures like the following, which may suggest, in a sense, the idea of a "straight angle."



It is convenient to have a name for a pair of angles such as $\angle BVA$ and $\angle BVC$ shown in the diagram.

DEFINITION. The two angles which are formed by three concurrent rays, two of which are opposite rays, are called a linear pair of angles.

Suppose we have a linear pair of angles, formed by a ray $\overrightarrow{VB}$ and two opposite rays $\overrightarrow{VA}$ and $\overrightarrow{VC}$, as shown in the preceding figure. We know, by the Protractor Postulate, that there is a ray-coordinate system in which

$\overrightarrow{VA}$ corresponds to 0 ,

and $\overrightarrow{VB}$ corresponds to a number between 0 and 180 , say b .

Then, by the definition of a ray-coordinate system,

$\overrightarrow{VC}$ corresponds to 180 ,

$m\angle AVB = b - 0 = b$,

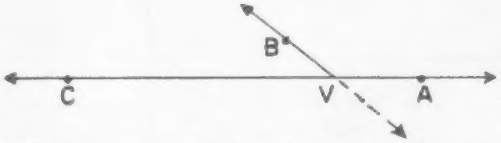
and $m\angle BVC = 180 - b$.

Therefore, $m\angle AVB + m\angle BVC = b + (180 - b) = 180$.

The importance of this result justifies stating it formally as a theorem.

THEOREM 4-8. The sum of the measures of the two angles in any linear pair is 180 .

The angles of a linear pair have one side in common. The line containing this common side separates the plane into two halfplanes. In the diagram, these two halfplanes are the side of $\overleftrightarrow{VB}$ containing A and the side of $\overleftrightarrow{VB}$ containing C .

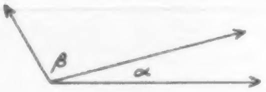


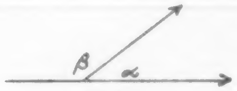
One of the halfplanes contains the interior of $\angle BVA$ and the other contains the interior of $\angle BVC$. Thus the interiors of the two angles of a linear pair do not intersect.

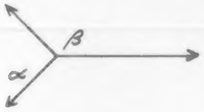
The next theorem may be proved by applying Theorems 4-4 and 4-8. The proof is left as a problem.

THEOREM 4-9. Let A, B, O, X, Y be distinct coplanar points such that O is between X and Y , such that A and B are on the same side of $\overleftrightarrow{XY}$, and such that $\overrightarrow{OA}$ is between $\overrightarrow{OX}$ and $\overrightarrow{OB}$. Then $m\,\angle XOA + m\,\angle AOB + m\,\angle BOY = 180$.

As a natural extension of the idea of a linear pair of angles, we have the concept of a pair of *adjacent angles*. In each of the following three diagrams, $\angle\alpha$ and $\angle\beta$ are adjacent angles.





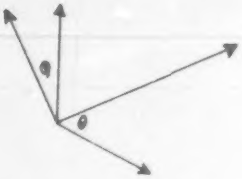


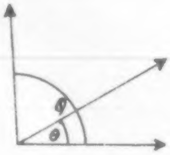
In each case, the two angles have one side in common. Moreover, their interiors do not intersect. These two ideas suggest the following definition.

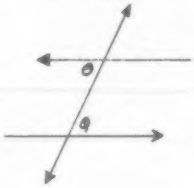
DEFINITION. Two coplanar angles are called a pair of adjacent angles if and only if they have one side in common and the intersection of their interiors is empty.

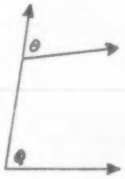
In particular, note that a linear pair of angles is also a pair of adjacent angles.

On the other hand, in each of the following four pictures, $\angle\theta$ and $\angle\alpha$ are not a pair of adjacent angles. In each case, tell which of the conditions in the definition fails to hold for the two angles.

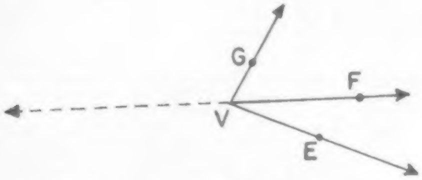








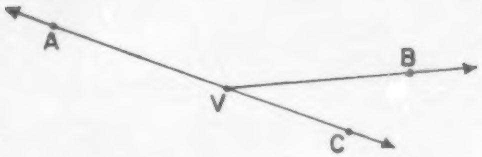
Let $\angle FVE$ and $\angle FVG$ be adjacent angles.



The line $\overleftrightarrow{VF}$ containing their common side separates the plane into two halfplanes. Since the interiors of the two angles do not intersect, they must lie on opposite sides of $\overleftrightarrow{VF}$. Likewise the interior of $\overrightarrow{VE}$ and the interior of $\overrightarrow{VG}$ lie on opposite sides of the line $\overrightarrow{VF}$.

THEOREM 4-10. Two adjacent angles, such that the sum of their measures is 180, are a linear pair of angles.

Proof: Let $\angle BVA$ and $\angle BVC$ be a pair of adjacent angles such that $m\angle BVA + m\angle BVC = 180$.



By the Protractor Postulate, there is a ray-coordinate system which assigns to $\overrightarrow{VB}$ the number 0 and assigns to $\overrightarrow{VA}$ a number less than 180, say a. Now C and A lie on opposite sides of $\overleftrightarrow{VB}$ and hence the ray-coordinate c of $\overrightarrow{VC}$ is greater than 180. Therefore, by the definition of a ray-coordinate system,

$$m\angle BVA = a - 0 = a,$$
$$m\angle BVC = 360 - (c - 0) = 360 - c.$$

By hypothesis, the sum of these measures is 180; that is,

$$a + (360 - c) = 180,$$

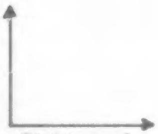
or

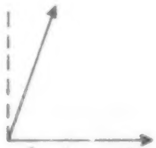
$$c - a = 180.$$

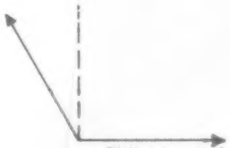
Hence $\overrightarrow{VC}$ and $\overrightarrow{VA}$ are opposite rays.

Because we decided to measure angles in degrees, the numbers 360 and 180 have had considerable significance in our development. We now begin our study of the importance of the number 90 in angle measurement.

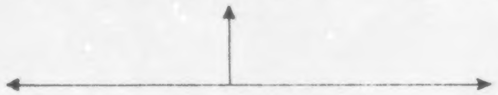
> DEFINITIONS. An angle whose measure is 90 is called a right angle. An angle whose measure is less than 90 is called an acute angle. An angle whose measure is greater than 90 is called an obtuse angle.


Right angle


Acute angle


Obtuse angle

THEOREM 4-11. If the two angles of a linear pair have the same measure, then each of them is a right angle.



Proof: Let r be the measure of each angle. By Theorem 4-8, the sum of their measures is 180. We have, then, $r + r = 180$, and hence $r = 90$. This shows that each of the angles is a right angle.

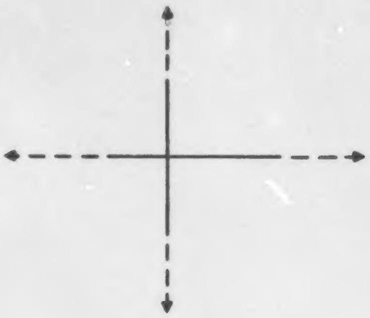
The concept of perpendicularity is a basic one in geometry, and is closely related to the notion of right angle. We first define perpendicular lines and then define perpendicularity for rays and segments.

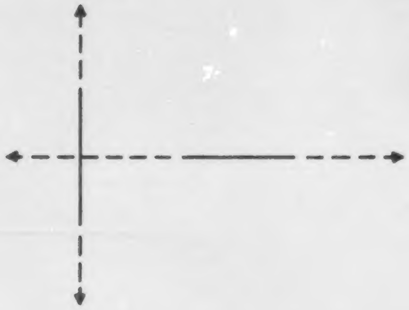
DEFINITION. The lines determined by two rays whose union is a right angle are called perpendicular lines.

Since every segment or ray determines a unique line which contains it, we have as a natural extension of the last definition,

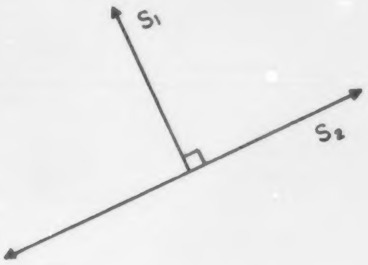
DEFINITION. Two sets each of which is a segment, a ray, or a line and which determine two perpendicular lines are called perpendicular sets, and each is said to be perpendicular to the other.

Since perpendicular sets, by definition, determine two intersecting lines, it is clear that they can intersect in at most one point and need not have any point in common. The following figure suggests these two possibilities in the case of two perpendicular segments.





Notation. We indicate that S_1 and S_2 are perpendicular sets by writing $S_1 \perp S_2$, or $S_2 \perp S_1$. In drawings we remind ourselves (when necessary) that two sets are perpendicular by the following symbolism.

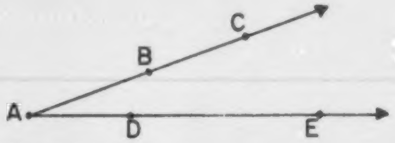


In Chapter 3 we agreed to call two segments *congruent* if they have the same length. We find the same idea suitable for angles.

DEFINITION. Two angles (whether distinct or not) which have the same measure are called *congruent angles*, and each is said to be *congruent to* the other.

Notation. We write $\angle ABC \cong \angle DEF$ to express the fact that $\angle ABC$ and $\angle DEF$ are congruent.

Although congruence and equality of angles seem to be very much alike, there are important differences between these concepts. For instance, in the next figure



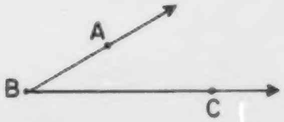
it is correct to write,

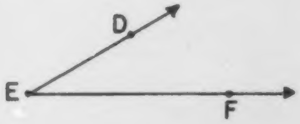
$$\angle BAD = \angle BAE = \angle DAB = \angle EAC = \ldots ,$$

because these are all names for the same angle. Furthermore, since this angle has a unique measure, it is also correct to write

$$\angle BAD \cong \angle BAE \cong \angle DAB \cong \angle EAC \cong \ldots .$$

Moreover, assuming that the angles shown in the following figure have the same measures, i.e., $m \angle ABC = m \angle DEF$,





it is correct to write

$$\angle ABC \cong \angle DEF .$$

However, since the angles are not the same, i.e., do not consist of the same points, it is not correct to write $\angle ABC = \angle DEF$.

THEOREM 4-12. Any two right angles are congruent to each other.

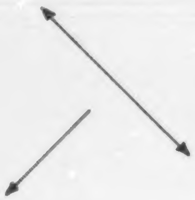
Proof: It follows from the definition of right angle that any two right angles have the same measure. It then follows from the definition of congruent angles that any two right angles are congruent.

Problem Set 4-8

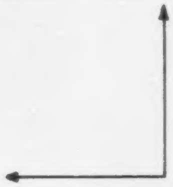
1. Tell which of the following five phrases correctly identifies the two perpendicular sets in each of the six diagrams:

   (a) two perpendicular rays,
   (b) two perpendicular segments,
   (c) a line perpendicular to a ray,
   (d) a ray perpendicular to a segment,
   (e) a segment perpendicular to a line.

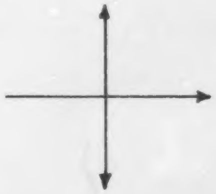
(1)



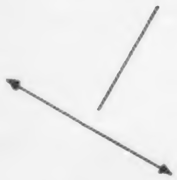
(2)



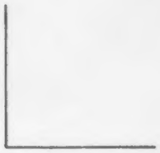
(3)



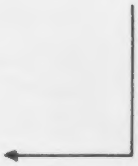
(4)



(5)



(6)

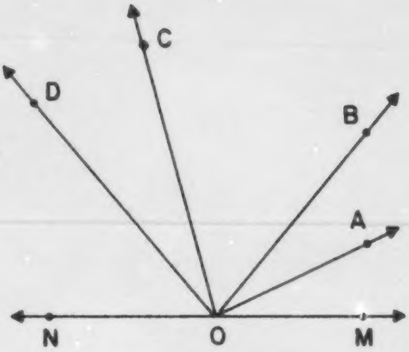


2. Two coplanar angles which have the following properties are called adjacent angles:

   (a) ________________ , and (b) ________________ .

3. In the accompanying figure,

   m $\angle DOM = 130$
   m $\angle MOB = 50$
   $\overrightarrow{OA}$ bisects $\angle BOM$
   m $\angle BOC$ = m $\angle COD + 30$



   Find the measure of each of the following:

   (a) $\angle AOM$ (d) $\angle COD$
   (b) $\angle AOB$ (e) $\angle BOC$
   (c) $\angle BOD$ (f) $\angle COA$

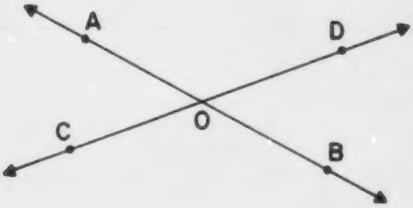
4. (a) Is every pair of adjacent angles a linear pair?
   (b) Is every linear pair of angles a pair of adjacent angles?

5. Complete each of the following sentences:

   (a) Each of the angles of a _______ pair which have the same measure is a _______ angle.
   (b) The lines determined by two rays which form a right angle are called _______ lines.
   (c) An angle whose measure is less than 90 is called an _______ angle.
   (d) An angle whose measure is more than 90 is called an _______ angle.
   (e) If one angle of a linear pair is acute, the other angle is _______ .

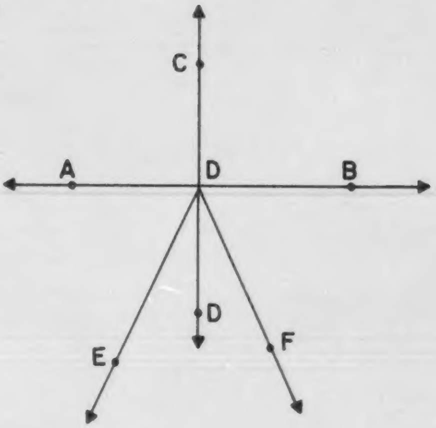
6. In the figure $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at O forming four angles. If m $\angle BOC = 128$, find:

   (a) m $\angle COA$
   (b) m $\angle AOD$
   (c) m $\angle BOD$



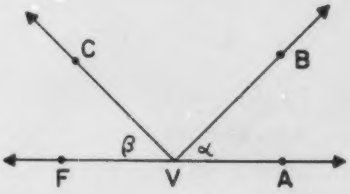
7. In the figure, $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$ and $\overrightarrow{OD}$ is the midray of $\angle EOF$ and $m\angle EOD = 25$. Find:

(a) $m\angle AOD$
(b) $m\angle DOB$
(c) $m\angle DOF$
(d) $m\angle AOE$
(e) $m\angle BOF$
(f) $m\angle COE$
(g) $m\angle COF$



8. In the figure, if $m\angle\alpha = 45$
$m\angle CVB = 90$
$m\angle\beta = 45$

which of the following statements are true in our development of geometry?



(a) $\angle\beta = \angle\alpha$.
(b) $\angle\beta \cong \angle\alpha$.
(c) $m\angle\beta = m\angle\alpha$.
(d) $\angle\alpha = \angle AVB$.
(e) $m\angle\alpha = \frac{1}{2}m\angle CVB$.
(f) $\angle CVA \cong \angle BVF$.
(g) $\angle\beta = \angle FVC$.
(h) $\angle AVB = \angle CVF$.

9. Which of the following expressions are meaningless in our development of geometry?

(a) $m\angle\alpha \cong m\angle\theta$.
(b) $m\angle PRN = 90$.
(c) $\angle\alpha = \frac{1}{2}\angle PRN$.
(d) $\angle\alpha + \angle PRN = \angle\beta + \angle PRN$.
(e) $\angle NQR = \angle PRM$.
(f) $\angle PRN \cong \angle PRN$.
(g) $\angle\theta + \angle\phi = 90$.
(h) $\angle PRN - \angle\alpha = \angle\theta$.

*10. Theorem 4-9 states: Let A, B, O, X, Y be distinct coplanar points such that O is between X and Y, such that A and B are on the same side of $\overleftrightarrow{XY}$, and such that $\overrightarrow{OA}$ is between $\overrightarrow{OX}$ and $\overrightarrow{OB}$. Then $m\angle XOA + m\angle AOB + m\angle BOY = 180$.

(a) Make a drawing to illustrate the situation described in the hypothesis.

(b) Give a proof of the theorem by combining the following two hints: with the aid of Theorem 4-4, show that $m\angle XOA + m\angle AOB = m\angle XOB$; with the aid of Theorem 4-8, show that $m\angle XOB + m\angle BOY = 180$.

(c) Give a proof of Theorem 4-9 which does not apply Theorems 4-4 and 4-8. Hint: Choose a ray-coordinate system in the plane relative to O such that $\overrightarrow{OX}$ is the zero-ray and $\overrightarrow{OB}$ has a ray-coordinate less than 180 ; express the measures of each of the angles $\angle XOA$, $\angle AOB$, $\angle BOY$ in terms of the ray-coordinates of $\overrightarrow{OX}$, $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OY}$, and then find the sum of the measures of the angles.

4-9. Supplements and Complements.

In the previous section we have considered two angles such that the sum of their measures is 180 . It is convenient to have names not only for this situation, but also for the case in which the sum is 90 instead of 180 .

DEFINITION. Two angles, the sum of whose measures is 180 , are called a pair of supplementary angles, and each is called a supplement of the other.

DEFINITION. Two angles, the sum of whose measures is 90 , are called a pair of complementary angles, and each is called a complement of the other.

The next theorem is a restatement of Theorem 4-8, using the new terminology.

THEOREM 4-13. (The Supplement Theorem) The two angles of any linear pair are supplementary to each other.

Although the angles of a linear pair are supplementary, it is not the case that every pair of supplementary angles is a linear pair. Thus Theorem 4-11 is a special case of the next theorem, whose proof resembles the proof of Theorem 4-11 and is left as a problem.

THEOREM 4-14. If two angles are both congruent and supplementary, then each of them is a right angle.

The following three theorems are all easy to prove. We give the proof of one, and leave the other proofs as problems.

THEOREM 4-15. If two angles are complementary, then each of them is acute.

THEOREM 4-16. Supplements of congruent angles are congruent to each other.

Proof: Let $\angle a$ and $\angle b$ be congruent angles, let $\angle c$ be any supplement of $\angle a$, and let $\angle d$ be any supplement of $\angle b$. We wish to show that $\angle c$ and $\angle d$ are congruent. We apply the definition of supplement twice and the definition of congruent angles:

$$\begin{aligned} m\angle a + m\angle c &= 180 \\ m\angle b + m\angle d &= 180 \\ m\angle a &= m\angle b \end{aligned}$$

We conclude that

$$m\angle c = m\angle d .$$

Thus $\angle c$ and $\angle d$ are congruent.

THEOREM 4-17. Complements of congruent angles are congruent to each other.

Problem Set 4-9

1. (a) Two angles, the sum of whose measures is 180, are called _______ angles, and each is called a _______ of the other.

   (b) Two angles, the sum of whose measures is 90, are called _______ angles, and each is called a _______ of the other.

2. Find the measure of the supplement of $\angle\alpha$ if $m\angle\alpha$ is equal to:

   (a) 110
   (b) 90
   (c) 36
   (d) 15.5
   (e) $75\frac{1}{4}$
   (f) n
   (g) $180 - n$
   (h) $90 - n$

3. Find the measure of the complement of $\angle\beta$ if $m\angle\beta$ is equal to:

   (a) 10
   (b) 80
   (c) 44.5
   (d) $37\frac{1}{2}$
   (e) x
   (f) $90 - x$
   (g) $180 - x$
   (h) $x + 45$

4. (a) Must angles which form a linear pair be adjacent? Must they be supplementary?

   (b) Must supplementary angles be adjacent? Must they form a linear pair?

   (c) Illustrate your answers in (b) by drawing diagrams.

5. If one of two supplementary angles has a measure which is 30 more than the measure of the other, what is the measure of each angle?

6. If the measure of an angle is twice the measure of its supplement, find the measure of the angle.

7. The measure of an angle is four times the measure of its supplement. Find the measure of each angle.

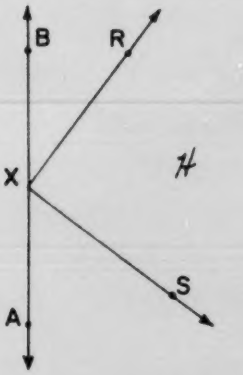
8. Find the measure of an angle which has twice the measure of its complement.

9. The measure of a supplement of an angle is equal to six times the measure of a complement of the given angle. Find the measure of the given angle.

10. The rays $\overrightarrow{XB}$ and $\overrightarrow{XA}$ are opposite rays in the edge of the halfplane $\mathcal{H}$; R and S are in $\mathcal{H}$; $m\angle RXB = 35$ and $m\angle RXS = 90$.

    (a) Name a pair of perpendicular rays, if any occur in the figure.

    (b) Name a pair of complementary angles, if any occur in the figure.



*11. If $\angle\alpha$ and $\angle\beta$ are two angles of a linear pair, then $m\angle\alpha + m\angle\beta =$ ______ , and $\angle\alpha$ and $\angle\beta$ are ________ angles. (This completes the proof of Theorem 4-13.)

12. If two angles of a linear pair have the same measure, then the measure of each angle is _________ .

*13. Theorem 4-14 states: If two angles are both congruent and supplementary, then each of them is a right angle.

Complete the following proof by filling in the blanks.

Let r be the measure of one of the given angles. Then the measure of the other given angle is also r , because the angles are ________ . The ________ of the measures of the two angles is 180 , because the two angles are _________ . Thus $r + r =$ ________ , and hence $r = 90$. Since each of the given angles has measure 90 , each of them is a ________ angle.

14. If two angles are both congruent and complementary, then the measure of each angle is __________ .

*15. If $\angle\alpha$ and $\angle\beta$ are complementary, then $m\angle\alpha + m\angle\beta =$ ______ . $m\angle\alpha > 0$; therefore, $m\angle\beta <$ ______ , and $\angle\beta$ is an ______ angle. $m\angle\beta > 0$; therefore, $m\angle\alpha <$ ______ , and $\angle\alpha$ is an ______ angle. (This completes the proof of Theorem 4-15.)

16. If $\angle\alpha$ and $\angle\beta$ are supplementary angles, and $\angle\alpha$ is acute, then $\angle\beta$ is ______ .

*17. Refer to the proof of Theorem 4-16 and use it as a model to prove Theorem 4-17.

*18. The rays $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ are concurrent in that order and $\overrightarrow{OA} \perp \overrightarrow{OC}$ and $\overrightarrow{OB} \perp \overrightarrow{OD}$. Answer each of the following:

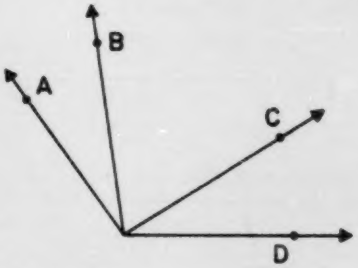
(a) $m\angle$ ______ + $m\angle$ ______ = $m\angle AOC$, and
$m\angle$ ______ + $m\angle$ ______ = $m\angle BOD$. Why?

(b) $\angle AOC$ and $\angle BOD$ are ______ angles. Why?

(c) $m\angle AOC =$ ______ , and
$m\angle BOD =$ ______ . Why?

(d) $\angle AOB$ is called a ______ of $\angle BOC$, and $\angle DOC$ is called a ______ of $\angle BOC$. Why?

(e) Therefore,
$\angle AOC$ ______ $\angle BOD$. Why?



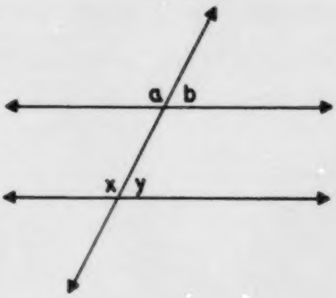
*19. Given in the figure to the right that $m\angle a = m\angle x$. Answer each of the following:

(a) $\angle a$ and $\angle b$ form a ______ pair, and $\angle x$ and $\angle y$ form a ______ pair.

(b) $\angle a$ and $\angle b$ are ______ angles, and $\angle x$ and $\angle y$ are ______ angles. Why?

(c) $\angle a \cong \angle$ ______. Why?

(d) $\angle$ ___ $\cong \angle$ ___ . Why?



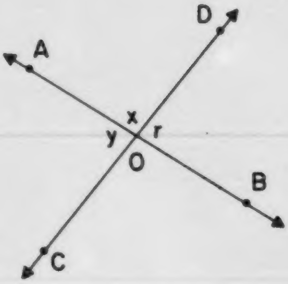
*20. $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at O.
Answer each of the following:

(a) $\angle y$ and $\angle x$ form a ______ pair, and $\angle$ ___ and $\angle r$ form a ______ pair. Why?

(b) $\angle$ ____ and $\angle$ ____ are supplements of $\angle x$. Why?

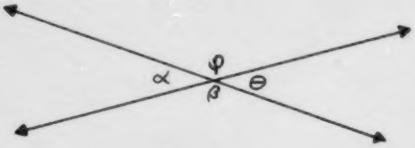
(c) $\angle x$ _____ $\angle x$. Why?

(d) $\angle$ ___ $\cong$ $\angle$ _____. Why?



4-10. Vertical Angles.

Two intersecting lines form four angles, like this:

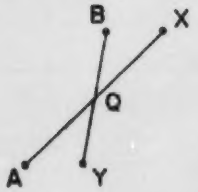


Pairs of angles which appear to be opposite, such as $\angle \alpha$ and $\angle \theta$, and $\angle \beta$ and $\angle Q$, are called vertical angles. More precisely we have the following definition.

DEFINITION. Two angles whose sides form two pairs of opposite rays are called a pair of vertical angles.

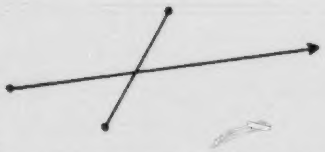
Are $\angle \alpha$ and $\angle \beta$ in the above diagram a pair of vertical angles? Why?

Suppose that two segments intersect at an interior point of each. For example, in the diagram, $\overline{AX}$ and $\overline{BY}$ intersect at Q.



These two segments determine two pairs of vertical angles. One of the pairs is $\angle AQY$ and $\angle BQX$. Name the other pair of vertical angles determined by the given segments.

If a ray and a segment intersect,



they determine two pairs of vertical angles.

These remarks are typical of the situations mentioned in the following theorem and also suggest the proof of the theorem.

THEOREM 4-18. Let each of two sets be a line or a ray or a segment. If the two lines which are determined, respectively, by the given sets intersect in a single point V , then the given sets determine two pairs of vertical angles, all with vertex V .

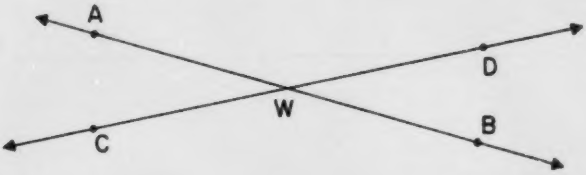
In Problem 1 of Problem Set 1-4, as an illustration of induction, we investigated vertical angles experimentally and we reached the tentative conclusion that the measures of vertical angles are equal. We are now in a position to deduce this statement from our postulates, thereby providing a logical as well as an experimental basis for accepting it.

The contrast between these two approaches is worthy of comment. In a study which made no use of deductive reasoning, every statement would have to be supported by experiment, and could be accepted with certainty only in the cases covered by the experiment. In a study which took no account of experimental results, the premises would have to be made up in arbitrary fashion, and the conclusions, except by coincidence, would bear no relation to the physical world. Our study of geometry is a blend of these two methods. Our postulates are suggested by experimentation; our theorems are obtained from them by deduction. And because our postulates are chosen to embody observations about the physical world, our theorems are not just interesting abstractions but are often useful results which can be checked with high accuracy in further experiments.

Stated formally, the result we wish to prove is the following.

THEOREM 4-19. Any two vertical angles are congruent to each other.

Proof: Let $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ be two lines which intersect in the point W to form the vertical angles, $\angle AWC$ and $\angle BWD$.



Now $\angle AWC$ and $\angle CWB$ are a linear pair of angles and hence are supplementary. (Why?) Also $\angle BWD$ and $\angle CWB$ are a linear pair of angles and are supplementary. Since each of $\angle AWC$ and $\angle BWD$ is a supplement of $\angle CWB$, Theorem 4-16 tells us that $\angle AWC$ and $\angle BWD$ are congruent to each other, as asserted.

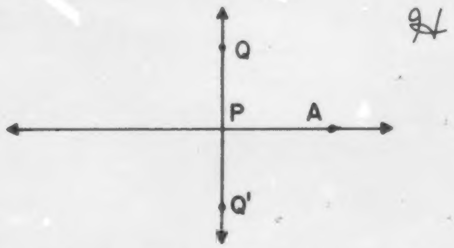
As an immediate consequence of this theorem, we have the following.

THEOREM 4-20. If two intersecting lines form one right angle, then they form four right angles.

By means of the previous theorem and the Angle Construction Theorem, we can prove the following useful result.

THEOREM 4-21. For each point on a line in a plane, there is one and only one line which lies in the given plane, contains the given point, and is perpendicular to the given line.

Proof: Let $\mathcal{H}$ be one of the halfplanes determined in the given plane (call it $\mathcal{F}$) by the given line (call it ℓ), and let A be any point of ℓ different from the given point (call it P). Then by the Angle Construction Theorem, there is a unique ray $\overrightarrow{PQ}$, with Q in $\mathcal{H}$ such that $\angle APQ$ is a right angle. Therefore $\overleftrightarrow{PQ}$ is perpendicular to ℓ and contains P . Similarly, there is a unique ray $\overrightarrow{PQ'}$, with Q' in the other halfplane of $\mathcal{F}$ determined by ℓ, such that $\angle APQ'$ is a right angle. Thus $\overleftrightarrow{PQ}$ and $\overleftrightarrow{PQ'}$ are the only lines in $\mathcal{F}$ perpendicular to ℓ and containing P . To complete our proof we must show that $\overleftrightarrow{PQ}$ and $\overleftrightarrow{PQ'}$ are the same. This follows from the facts that $\angle APQ$ and $\angle APQ'$ are adjacent (Why?) and supplementary (Why?), and therefore, by Theorem 4-10, $\overrightarrow{PQ}$ and $\overrightarrow{PQ'}$ are opposite rays.

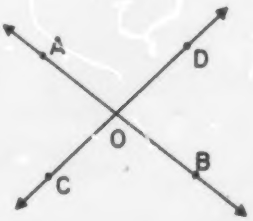


In this theorem it is important that all lines considered should lie in a single plane. If this restriction is eliminated, how many lines are perpendicular to a given line at a given point? What do you think is the set of all points which lie on such lines?

If we eliminate the requirement that the perpendicular to ℓ must pass through a particular point, P , how many lines are there in $\mathcal{F}$ which are perpendicular to ℓ ? What do you think is the set of all points which lie on such lines? Finally, if we eliminate both the requirement that the perpendicular lie in $\mathcal{F}$ and the requirement that it contains a particular point P , how many lines are there in space which are perpendicular to ℓ ? What do you think is the set of all points which lie on such lines?

The related question, about the existence and uniqueness of a line perpendicular to ℓ and containing a given point not on ℓ , is not covered by the preceding theorem. We shall discuss it after we develop the idea of congruent triangles.

Problem Set 4-10



*1. In the figure if $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at O and m $\angle AOC = 90$, then

(a) m $\angle BOD$ = _____ because $\angle AOC$ and $\angle BOD$ are _____ angles.

(b) m $\angle AOD$ = _____ because $\angle AOC$ and $\angle BOD$ form a _____ pair.

(c) m $\angle BOC$ = _____ because $\angle AOD$ and $\angle BOC$ are _____ angles.

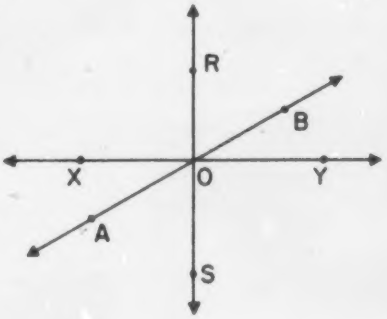
(This completes the proof of Theorem 4-20.)

2. In the figure $\overleftrightarrow{AB}$, $\overleftrightarrow{RS}$, and $\overleftrightarrow{XY}$ are coplanar and intersect at O . $\overleftrightarrow{RS} \perp \overleftrightarrow{XY}$. Name four pairs of:

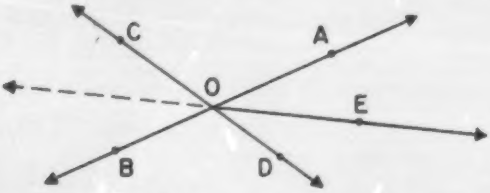
(a) vertical angles.

(b) supplementary angles.

(c) complementary angles.



3. Two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at point O and $\overrightarrow{OE}$ is the midray of $\angle AOD$.



(a) Suppose that $m\ \angle AOD = 50$. Show that the ray opposite to $\overrightarrow{OE}$ is the midray of $\angle COB$.

(b) Suppose that $m\ \angle AOD = 80$. Is the conclusion in (a) true in this case?

(c) Justify the conclusion in (a) in the case that $m\ \angle AOD = x$.

(d) State the conclusion in words, without reference to the symbols in the diagram.

*4. Let the ray-coordinates of three coplanar rays $\overrightarrow{VR}$, $\overrightarrow{VS}$, $\overrightarrow{VT}$ be 0 , 100 , 240 , respectively.

(a) Draw a diagram, showing the three rays.

(b) Show, on your diagram, the interior of each of the three angles $\angle RVS$, $\angle SVT$, $\angle TVR$.

(c) Among the three angles, do any two of them have interiors which intersect?

(d) Find the measure of each of the three angles.

(e) Find the sum of the measures of the three angles.

*5. Answer the same questions as in Problem 4 , for the case in which the ray-coordinates of $\overrightarrow{VR}$, $\overrightarrow{VS}$, $\overrightarrow{VT}$ are 0 , 70 , 270 , respectively.

*6. Complete the proof of the following result by filling in the blanks. (It will be helpful to your understanding if you draw your own diagram.)

If $\angle AVB$, $\angle BVC$, $\angle CVA$ are three coplanar angles such that the interiors of any two of them do not intersect, then the sum of the measures of the three angles is 360 .

Proof: In the plane containing the angles, there is, by Postulate ___ , a ray-coordinate system such that $\overrightarrow{VA}$ is the zero-ray and such that $\overrightarrow{VB}$ has a ray-coordinate, say b , which is less than _____ . By _______, the interiors of $\angle AVB$ and $\angle AVC$ do not intersect; therefore, C and B lie on _______ (same side, or opposite sides) of $\overleftrightarrow{VA}$. Consequently the ray-coordinate of $\overrightarrow{VC}$, say c , is ________ than 180 . By _______, the interiors of $\angle BVC$ and $\angle BVA$ do not _____; therefore, $\overrightarrow{VA}$ is not between $\overrightarrow{VB}$ and $\overrightarrow{VC}$. Consequently, $c - b < 180$. By the definition of a ______,

$$m\angle AVB = b - 0 = b ,$$
$$m\angle BVC = c - b ,$$
and
$$m\angle CVA = \_\_\_\_\_ .$$

The sum of these measures is

$$m\angle AVB + m\angle BVC + m\angle CVA = b + (c - b) + \_\_\_\_\_ = 360 .$$

7. Let a ray-coordinate system in a plane assign numbers 0 , 60 , 200 , 320 to the rays $\overleftrightarrow{VE}$, $\overleftrightarrow{VF}$, $\overrightarrow{VG}$, $\overrightarrow{VH}$, respectively.

   (a) Draw a diagram, showing the four rays.

   (b) Show, on your diagram, the interiors of each of the four angles $\angle EVF$, $\angle FVG$, $\angle GVH$, $\angle HVE$.

   (c) Among these four angles, do any two of them have interiors which intersect?

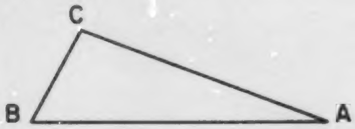
   (d) Find the measure of each of the four angles.

   (e) Find the sum of the measures of the four angles.

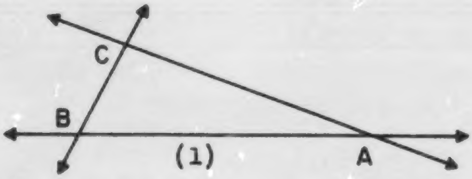
8. Answer the same questions as in Problem 7 , for the case in which the ray-coordinates of $\overrightarrow{VE}$, $\overrightarrow{VF}$, $\overrightarrow{VG}$, $\overrightarrow{VH}$ are 0 , 60 , 100 , 320 , respectively.

4-11. Triangles and Quadrilaterals.

We all know what a triangle looks like. We also know that a triangle has three angles. If we draw a triangle we make a figure which looks like this.



If we now draw the three angles of this same triangle our figure looks like this.



This figure is certainly a drawing of three angles but is not a drawing of what we usually mean by a triangle. Try to define "triangle" so that the objects you specify correspond to pictures which look like what we usually mean by triangles.

> *DEFINITION*. The union of the three segments determined by three noncollinear points is called a *triangle*.

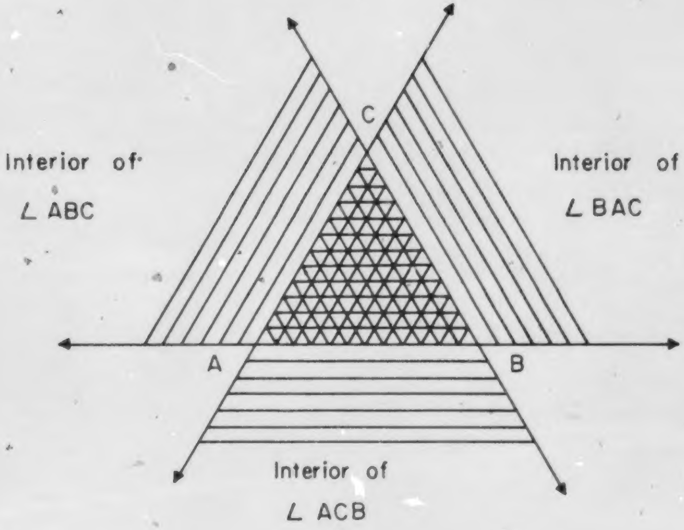
> *Notation*. If A, B, C are three noncollinear points, the triangle which they determine is denoted by the symbol $\triangle ABC$.

> *DEFINITIONS*. Each of the points A, B, C is called a *vertex* of $\triangle ABC$; each of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ is called a *side* of $\triangle ABC$; each of the angles $\angle ABC$, $\angle BCA$, $\angle CAB$ is called an *angle* of $\triangle ABC$. A side of a triangle and the angle whose vertex is not a point of that side are said to be *opposite* each other.

We must be careful to observe that whereas the vertices and sides of a triangle are actually contained in the set of points which constitutes the triangle, the angles of a triangle are not contained in the triangle. In fact each angle of a triangle is a set of points formed by two rays, neither of which is contained in the triangle (Figure 1). Thus, while it is quite proper to speak of "the angles of a triangle" or "the angles determined by a triangle," it is incorrect to speak of "the angles contained in a triangle." Perhaps it will make the distinction clear if we observe that whereas a school has alumni, or determines a set of alumni, the school does not contain its alumni.

DEFINITION. The intersection of the interiors of the three angles of a triangle is called the interior of the triangle.

The following figure illustrates how the interior of a triangle is the intersection of the interiors of the three angles of the triangle.

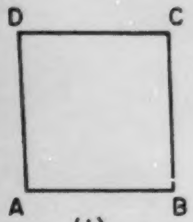


DEFINITION. The set of all those points in the plane of a triangle which are not contained in the triangle nor in its interior is called the *exterior of the triangle*.

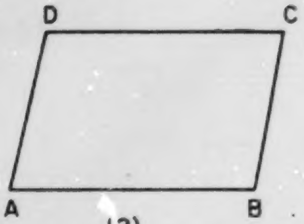
THEOREM 4-22. The interior of a triangle is a convex set.

Proof: It is the intersection of convex sets.

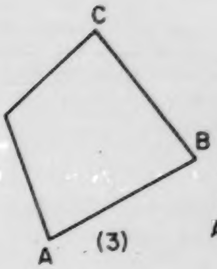
As natural extensions of the idea of a triangle, we have the concept of *quadrilaterals* and *polygons* in general. By a quadrilateral we mean, roughly, a four-sided figure like one of the following:



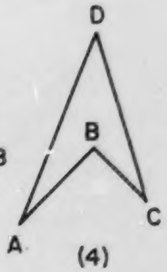
(1)



(2)

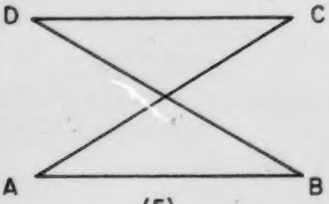


(3)



(4)

but not like

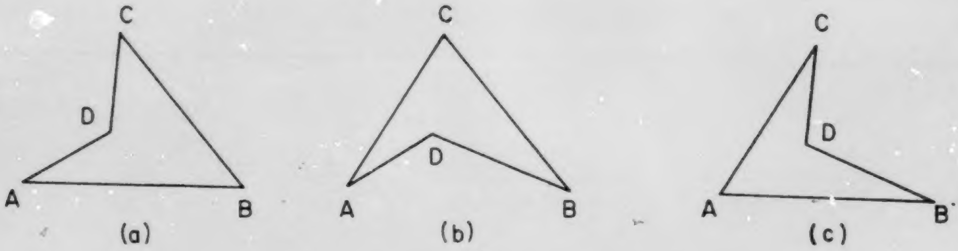


(5)

What difference can you see between the last figure and each of the others? Using these figures as guides, write your own definition of quadrilateral.

DEFINITIONS. Let A, B, C, D be coplanar points such that no three are collinear, and such that none of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ intersects any other at a point which is not one of its endpoints. Then we call the union of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ a quadrilateral, we call each of the four segments a side of the quadrilateral, and we call each of the points A, B, C, D a vertex of the quadrilateral.

Is it possible for a single set of four points to be the vertices of different quadrilaterals? The following figure shows that it is possible.



Notice that in this figure the same set of vertices occurs in each quadrilateral, but not the same set of sides. It thus appears that a quadrilateral is not determined by its vertices, and cannot be described by a list of the vertices unless the list also indicates the sides of a quadrilateral. We therefore agree that when we list the vertices of a quadrilateral we shall do it in such a way that

(a) vertices which are adjacent in the list are endpoints of the same side,

(b) the first and last vertices in the list are endpoints of the same side.

Notation. We shall refer to the quadrilateral whose vertices are A, B, C, D and whose sides are $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ as the quadrilateral ABCD .

In this notation, the quadrilateral shown in Figure (a), above, can properly be referred to as

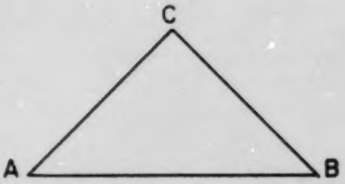
quadrilateral ABCD , or
quadrilateral BCDA , or
quadrilateral CDAB , or
quadrilateral BADC ,

but not as quadrilateral ACDB . Why? For which of the three figures shown above is quadrilateral ACDB a suitable name?

Problem Set 4-11

1. Complete this definition of a triangle: a triangle is the _____ of the three _____ determined by three _____ points.

2. Are the sides $\overline{AC}$ and $\overline{AB}$ of $\triangle ABC$ the same as the sides of $\angle A$? Explain.



3. Is the union of two of the angles of a triangle the same as the triangle itself? Why?

4. Is a triangle a convex set? Why?

5. Is the exterior of a triangle a convex set?

6. (a) Can a point be in the exterior of a triangle and in the interior of an angle of the triangle? Illustrate.
   (b) Can a point be in the exterior of a triangle and not in the interior of any angle of the triangle? Illustrate.

7. Given $\triangle ABC$, and a point P which is in the interior of $\angle BAC$ and also in the interior of $\angle ACB$. What can you conclude about point P ?

8. Given $\triangle ABC$ and a point P such that P and A are on the same side of $\overleftrightarrow{BC}$ and such that P and B are on the same side of $\overleftrightarrow{AC}$.

   (a) Is P in the interior of $\angle ACB$?

   (b) Is P in the interior of $\triangle ABC$?

9. Illustrate the fact that the exterior of a triangle is the same as the union of the exteriors of the three angles of the triangle.

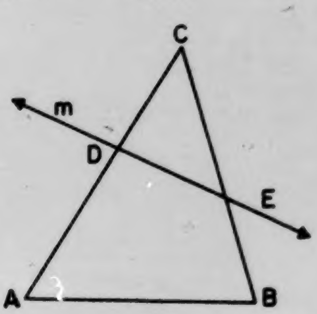
10. ABCD names a quadrilateral. Which of the following names denote this same quadrilateral?

(a)	ACBD	(e)	DABC
(b)	ADBC	(f)	CDBA
(c)	BCDA	(g)	DBCA
(d)	BACD	(h)	CBAD

11. Sketch a picture showing four distinct coplanar points such that the union of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ is not a quadrilateral. Is there more than one possibility? Explain.

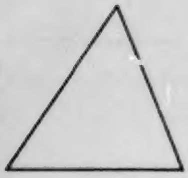
12. Explain why the following statement is true:

    If a line m intersects two sides of a triangle ABC in points D and E , not the vertices of the triangle, then line m does not intersect the third side. (Hint: Show that A and B are in the same halfplane with edge m .)

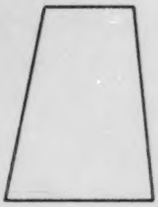


4-12. Polygons.

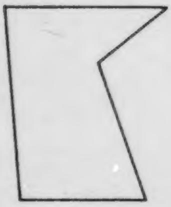
A polygon is a figure like this:



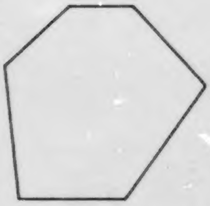
(1)



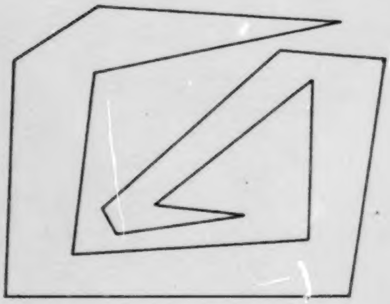
(2)



(3)

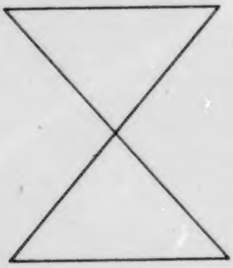


(4)



(5)

But not like this:

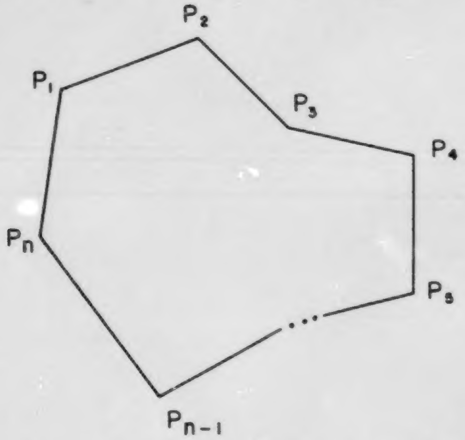


Triangles and quadrilaterals are special kinds of polygons.

The idea of a polygon can be expressed more precisely as follows. Suppose that we have given a sequence

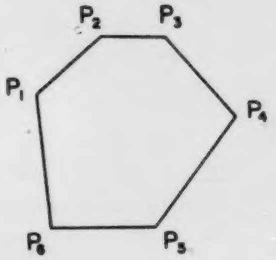
$$P_1 , P_2 , \ldots , P_n$$

of distinct points in a plane. We join each point to the next one by a segment, and finally we join P_n to P_1 .

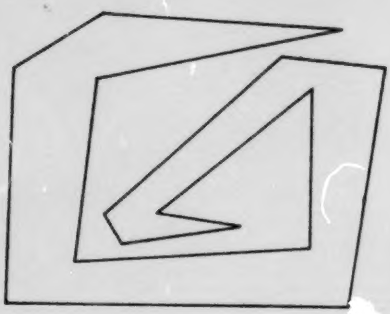


In the figure, the dots indicate other possible points and segments, because we do not know how large n is. Notice that the point just before P_n is P_{n-1} , as it should be.

As an example, the set shown in Figure (4) above is the union of six segments, as pictured below.



Check that, if we label each corner point, or vertex, of the set which was shown in Figure (5) above and is reproduced below,



we would have fifteen points P_1 , P_2 , ... , P_{15} , and the set is the union of fifteen segments joining certain pairs of these points.

DEFINITIONS. Let P_1 , P_2 , P_3 , ... , P_{n-1} , P_n $(n \geq 3)$ be n distinct points in a plane. Let the n segments $\overline{P_1P_2}$, $\overline{P_2P_3}$, ... , $\overline{P_{n-1}P_n}$, $\overline{P_nP_1}$ have the properties:

(1) no two segments intersect except at their endpoints, and

(2) no two segments with a common endpoint are collinear.

Then the union of the n segments is a polygon. Each of the n given points is called a vertex of the polygon. Each of the n segments is called a side of the polygon.

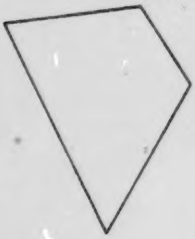
If n = 3 , this is precisely the definition of a triangle; and if n = 4 , it is the definition of a quadrilateral. For a few more values of n we also have special names which are commonly used and which you should learn:

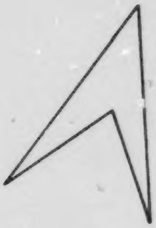
Value of n	Name of polygon with n sides
3	triangle
4	quadrilateral
5	pentagon
6	hexagon
7	heptagon
8	octagon
9	nonagon
10	decagon
12	dodecagon

Notation. We shall refer to the polygon whose vertices are P_1 . P_2 , ... , P_n and whose sides are $\overline{P_1P_2}$, $\overline{P_2P_3}$, ... , $\overline{P_nP_1}$ as the polygon $P_1P_2 \ldots P_n$.

DEFINITIONS. Two vertices of a polygon which are endpoints of the same side are called consecutive vertices. Two sides of a polygon which have a common endpoint are called consecutive sides.

Much of our discussion of polygons will be restricted to a special type of polygon, whose description involves the notion of a convex set. Most of us would say that, in the following pictures, the "interior" of the first quadrilateral is a convex set while the "interior" of the second is not.

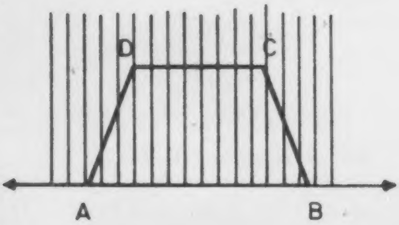


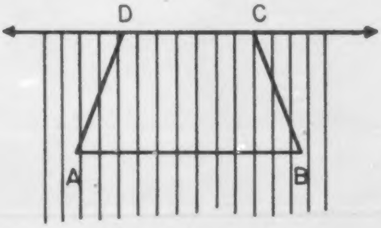


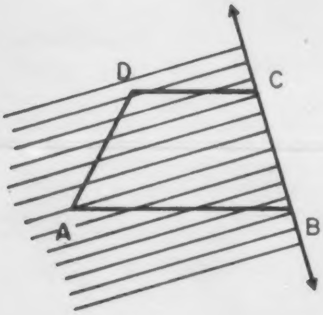
Try to give a definition which expresses what you understand by a convex polygon.

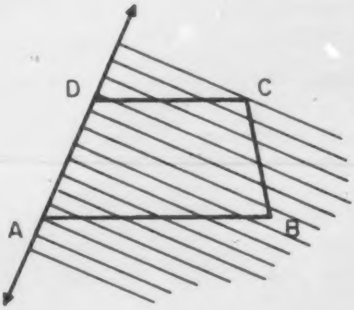
DEFINITION. A polygon is a convex polygon if and only if each of its sides lies in the edge of a halfplane which contains all the rest of the polygon.

The following figures illustrate how each side of a convex quadrilateral lies in the edge of a halfplane which contains all the rest of the quadrilateral.

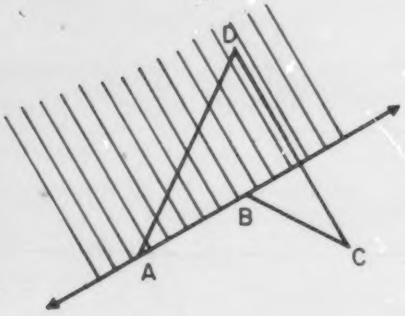


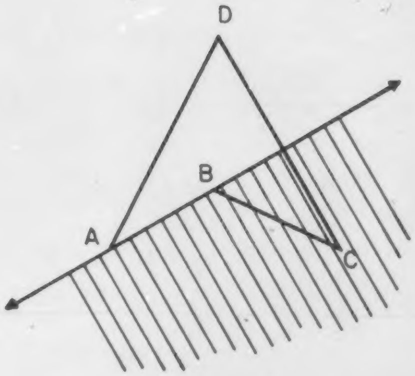






The following figures illustrate that for a quadrilateral which is not convex this is not the case.





DEFINITION. The interior of a convex polygon is the intersection of all the halfplanes each of which has a side of the polygon on its edge and contains the rest of the polygon.

Notice that every triangle is a convex triangle and that the interior of a convex polygon with three sides, as defined here, is the same set as the interior of the triangle, as defined in the preceding section.

Observe carefully that a convex polygon is not itself a convex set. (Why?) The following theorem explains what is convex about a convex polygon.

THEOREM 4-23. The interior of any convex polygon is a convex set.

Proof: It is the intersection of halfplanes, and each of them is a convex set.

DEFINITION. Any segment which joins two vertices of a convex polygon and is not a side of the polygon is called a diagonal of the convex polygon.

DEFINITIONS. Any angle determined by a pair of consecutive sides of a convex polygon is called an *angle of the convex polygon*. Two angles of a convex polygon are called *consecutive angles* if and only if their intersection is a side of the polygon.

Note that any two vertices of a triangle are consecutive, any two sides of a triangle are consecutive, and any two angles of a triangle are consecutive. If two vertices of a polygon, or two sides of a polygon, or two angles of a convex polygon are not consecutive, we may easily describe them as being "nonconsecutive." However, for the case of the quadrilateral, the special word, "opposite," is frequently used.

DEFINITIONS. Two vertices of a quadrilateral which are not consecutive are called *opposite vertices*, and each is said to be *opposite to* the other. Two sides of a quadrilateral which are not consecutive are called *opposite sides*, and each is said to be *opposite to* the other. Two angles of a convex quadrilateral which are not consecutive are called *opposite angles*, and each is said to be *opposite to* the other.

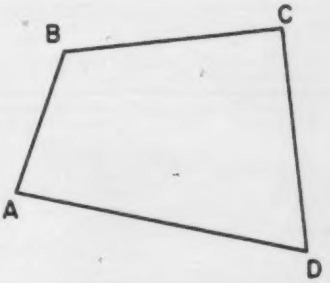
Given a side of a quadrilateral, there is exactly one side of the quadrilateral which is opposite to it. Similarly each vertex of a quadrilateral determines an opposite vertex. Likewise, each angle of a convex quadrilateral has a unique angle opposite it. Do you see why we do not speak in general about the "opposite sides of a polygon" or the "opposite vertices of a polygon" or the "opposite angles of a convex polygon"?

Problem Set 4-12

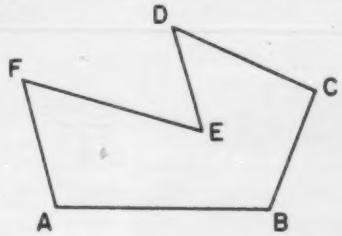
1. Given the polygon ABCDEF , which of the following names denote the same polygon?

   (a) CDAEBF
   (b) DEFABC
   (c) FBADEC
   (d) FEDCAB
   (e) BAFEDC
   (f) DEFACB
   (g) EBFCAD
   (h) DEBACF
   (i) EDCBAF
   (j) BCFEDA

2. In this quadrilateral ABCD , $\overline{BC}$ is called a _____ of the quadrilateral, $\overline{BD}$ is known as a _____ , $\overline{BC}$ and $\overline{AD}$ are _______ sides, $\overline{BC}$ and $\overline{CD}$ are _______ sides, $\angle C$ and $\angle D$ are called _____ angles, $\angle B$ and $\angle D$ are _____ angles, and point A is a _______ .



3. Is ABCDEF a picture of a convex polygon? Explain your answer.



4. (a) Draw a picture of a convex polygon of each of the following types:

   Triangle
   Quadrilateral
   Pentagon
   Octagon
   Decagon

   (b) Choose a single vertex in each polygon and draw all the diagonals from that vertex.

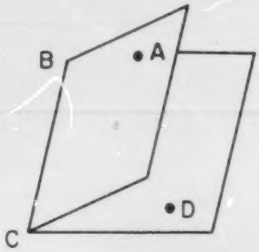
   (c) Compare, in each case, the number of diagonals with the number of sides. What generalization seems to be true?

4-13. Dihedral Angles.

The idea of an angle as the union of two concurrent rays is essentially a concept of plane geometry, since no matter how the angle be oriented in space, its two rays will always determine a unique plane (Why?) which will contain the angle. As a natural generalization of this concept from plane to space geometry, we have the idea of a dihedral angle.

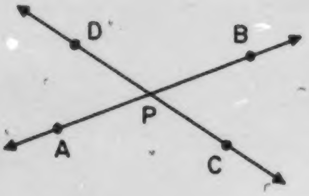
> DEFINITIONS. A dihedral angle is the union of a line and two halfplanes having this line as edge and not lying in the same plane. The line is called the edge of the dihedral angle. The union of the line and either halfplane is called a face of the dihedral angle.

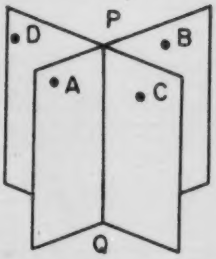
A dihedral angle is thus a figure like that suggested by two pages of an open book or two intersecting walls of a room.



> Notation. If $\overleftrightarrow{BC}$ is the edge of a dihedral angle and if A and D are points in different faces of the angle but not on the edge, the angle is denoted by the symbol ∠A-BC-D .

Clearly, two intersecting planes form four dihedral angles, just as two intersecting lines form four (plane) angles:

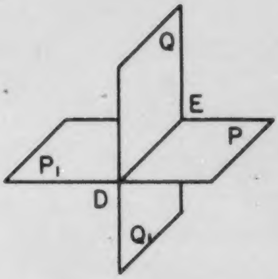




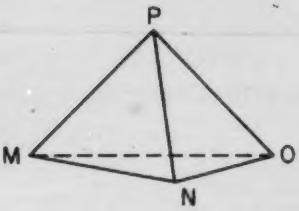
We shall speak of pairs of dihedral angles such as $\angle$A-PQ-C and $\angle$B-PQ-D as vertical dihedral angles. (Which dihedral angle forms with $\angle$A-PQ-D a pair of vertical dihedral angles?). We postpone all discussion of the measure of dihedral angles until we have studied lines and planes more extensively.

Problem Set 4-13

1. Name the two pairs of vertical dihedral angles in the figure.



2. This pyramid with a triangular base (called a tetrahedron) has six dihedral angles. Name them.



3. Describe some dihedral angles which can be seen in your classroom.

4-14. Summary.

An angle is the union of two noncollinear, concurrent rays. With the agreement that the unit for measuring angles is the degree, every angle has a unique measure. The measure is a real number between 0 and 180 . An angle may be classified as an acute angle, a right angle, or an obtuse angle according as its measure is less than, equal to, or greater than half of 180 . The sides of a right angle determine two lines which form four angles all with the same measure. The particular importance of the right angle will gradually be revealed as we observe the many situations in which two perpendicular sets play a role.

We often consider a pair of angles. Several useful types are a pair of supplementary angles, a pair of complementary angles, a linear pair of angles, a pair of adjacent angles, a pair of vertical angles. The condition that two angles be supplementary or be complementary depends only on the measures of the angles. The condition that two angles be a linear pair, or be adjacent angles, or be vertical angles insists (among other requirements) that the two angles be coplanar and have a common vertex.

In Chapter 3 we studied a coordinate system on a line, that is, a one-to-one correspondence which matches points and numbers such that the distance between points is related to the difference between numbers. In Chapter 4 we studied a ray-coordinate system in a plane relative to a point, that is, a one-to-one correspondence which matches rays and numbers such that the measure of an angle formed by rays is related to the difference between numbers. The Ruler Postulate in Chapter 3 gives us coordinate systems on a line. The Protractor Postulate in Chapter 4 gives us ray-coordinate systems in a plane. Furthermore each of these postulates states that, if certain requirements are satisfied, the system is unique. A coordinate system on a line enables us to describe betweenness for three distinct collinear points. A ray-coordinate system in a plane enables us to describe betweenness for three distinct concurrent rays, whose interiors lie in a halfplane. The Betweenness-Distance Theorem relates betweenness for points and three distances; the Betweenness-Angles Theorem relates betweenness for rays and the measures of three angles. A segment has an interior consisting of all points between its endpoints. An angle has an interior consisting of all interior points of rays between its sides.

In a given plane, any line separates the plane into two halfplanes, any angle separates the plane into the interior and the exterior of the angle, any triangle separates the plane into the interior and the exterior of the triangle, and (more generally) any convex polygon separates the plane into the interior and the exterior of the convex polygon. Every halfplane, the interior of every angle, the interior of every triangle, and the interior of every convex polygon are convex sets.

In the preceding chapter we introduced congruence of segments. In the present chapter we introduced congruence of angles. In the next chapter we shall introduce congruence of triangles. We shall study congruence in much greater detail and learn more about proving theorems.

Vocabulary List

- convex set
- halfplane
- angle
- measure of an angle
- ray-coordinate system
- betweenness (for rays)
- midray
- interior (of an angle)
- linear pair of angles
- adjacent angles
- right angle
- acute angle
- obtuse angle
- vertical angles
- triangle
- quadrilateral
- polygon
- convex polygon
- dihedral angle
- perpendicular
- congruent angles
- supplementary angles
- complementary angles

Review Problems

1. Fill in the blanks:

   (a) The set of all points in a plane which lie on one side of a given line of the plane is called a ______________ .

   (b) Two sets of points into which a ___________ separates space are called halfspaces.

   (c) The line which separates a plane into two halfplanes is called the _________ of each halfplane.

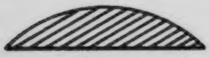
   (d) The set of all points in a halfplane is a ____________ set.

   (e) The intersection of a halfplane and its edge is ______________ .

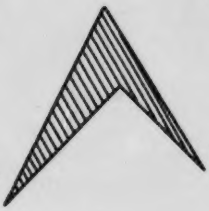
2. (a) Given the set of all points on a line whose coordinates satisfy $3 < x \leq 10$. Is this set of points a convex set?

   (b) Tell whether the set which is suggested by shading in each of the following pictures is a convex set.

(1)



(2)



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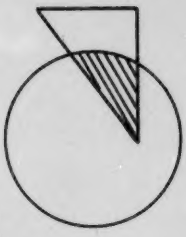


3. In a given plane the ray-coordinates assigned to $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$, and $\overrightarrow{VD}$ are, respectively, 0 , 25 , 105 , and 280 . Find the measure of each of the following angles.

   (a) m $\angle BVC$
   (b) m $\angle AVD$
   (c) m $\angle DVB$
   (d) m $\angle CVD$
   (e) m $\angle AVC$

4. Fill in the blanks in each of the following statements:

(a) The ray that _______ an angle is called the midray of the angle.

(b) To every angle there corresponds a unique number k (_____ $< k <$ _____) , called the _______ of the angle.

(c) An angle whose measure is less than 90 is called _______.

(d) An angle whose measure is greater than 90 is called _______.

(e) Angles with the same measure are _______.

(f) If two angles are both congruent and supplementary, then each of them is a _______.

(g) Supplements of congruent angles are _______.

(h) If two angles are complementary, then each of them is _______.

(i) An angle is the _______ of two _______ which have a common endpoint and do not lie in the same line.

(j) If X, Y, Z are three noncollinear points, the union of the three segments determined by these points is a _______.

(k) If X, Y, Z are distinct collinear points, the union of the three segments determined by these points is a _______.

(l) If one of two supplementary angles has a measure which is 50 more than the measure of the other, then the measures of the two angles are _______ and _______.

(m) The measure of an angle is five times the measure of its complement. The measures of the two angles are _______ and _______.

(n) If the angles of a linear pair are congruent, each of the angles is a _______.

5. Is there a point in the plane of a triangle such that the point is neither in the exterior nor the interior of the triangle and neither in the interior nor the exterior of any of its angles? Explain.

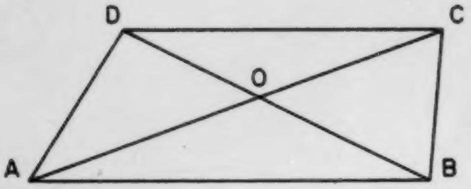
6. Is the measure of one angle added to the measure of another angle necessarily the measure of an angle? Explain.

7. In the following figure

$m\angle BCD = 90$,
$m\angle DAO = 45$,
$m\angle BOC = 50$,
$m\angle DCO = 25$.

Find:

(a) $m\angle DOC$.
(b) $m\angle BCO$.
(c) $m\angle DOA$.
(d) $m\angle AOB$.



Which one of the given data is unnecessary?

8. Given $\angle RST$ and point X between R and T , is X in the interior of the angle? Why?

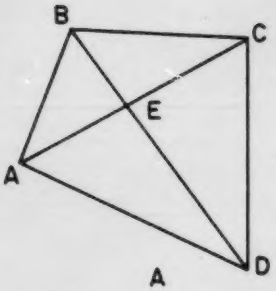
9. If the measure of an angle in degrees is 63 , what is its measure in right-angles? In grads?

10. Given noncollinear rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ with ray-coordinates b and c , respectively, what is $m\angle BAC$? Discuss four possible cases.

11. If B, C, D are points in a halfplane, $\mathcal{H}$, whose edge contains the point A , and if $\overrightarrow{AC}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AD}$, state the relation between the measures of the three angles determined by the three rays.

12. Can the interior of a triangle be considered as the intersection of three halfplanes? Illustrate.

13. Refer to the figure at the right to answer the following questions.



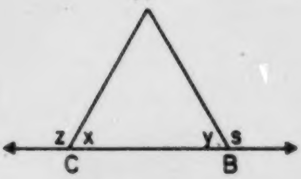
(a) How many triangles are determined by A, B, C, D, E in this figure?

(b) Does m $\angle BAC$ = m $\angle BAE$?

(c) Does $\angle BAC = \angle BAE$?

(d) Is $\angle ABE$ supplementary to $\angle EBC$?

14. If in the figure $\angle x \cong \angle y$, explain why $\angle z \cong \angle s$.



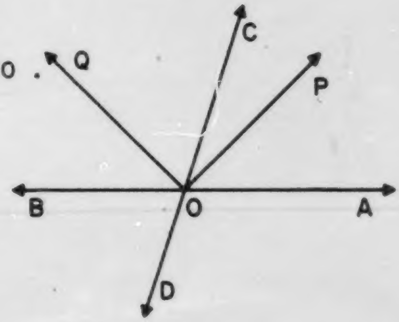
15. In this figure, $\overleftrightarrow{CD}$ and $\overleftrightarrow{AB}$ intersect at O ; $\overrightarrow{OP}$ is a midray of $\angle AOC$; $\overrightarrow{OQ}$ is a midray of $\angle COB$; m $\angle AOC$ = 70 . Find:

(a) m $\angle COB$

(b) m $\angle POC$

(c) m $\angle COQ$

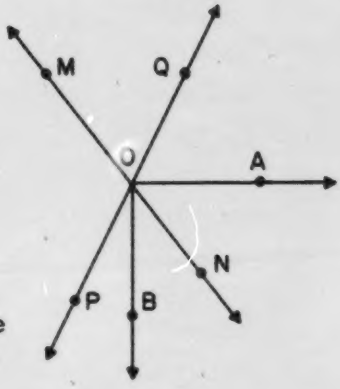
(d) m $\angle POQ$



16. In the figure, $\overleftrightarrow{PQ}$ and $\overleftrightarrow{MN}$ intersect at O , $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are midrays of $\angle QON$ and $\angle PON$, respectively.

(a) m $\angle AOB$ = __________.
Justify your answer.
Hint: Let m $\angle PON = x$.

(b) Does this prove that the union of the midrays of the angles of a linear pair is a right angle?



17. Is the following a correct restatement of the Angle Construction Theorem: Given a ray $\overrightarrow{XY}$ and a number k between 0 and 180 there is exactly one ray $\overrightarrow{XP}$ such that $m\angle PXY = k$? Explain.

18. Is the following statement true in every case? If $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at O , then $\angle AOC \cong \angle BOD$.

19. In a ray-coordinate system let the ray-coordinates of $\overrightarrow{AD}$, $\overrightarrow{AP}$, $\overrightarrow{AE}$ be d, p, e, respectively. Suppose that $\overrightarrow{AP}$ is between $\overrightarrow{AD}$ and $\overrightarrow{AE}$ and that $\frac{m\angle DAP}{m\angle DAE} = \frac{1}{2}$. Express p in terms of d and e in each of the following three cases.

    (a) $0 < e - d < 180$.
    (b) $p < d < 90$ and $e > 270$.
    (c) $d < 90$ and $270 < e < p$.

20. In triangle ABC , let D be the midpoint of side $\overline{BC}$, and let P be a point on ray $\overrightarrow{AD}$. Is P in the interior of $\angle BAC$? Is P in the interior of $\triangle BAC$? Justify your answers.

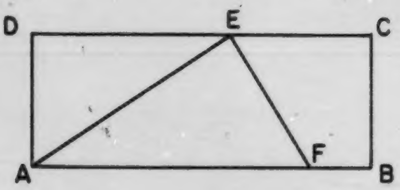
21. What can be asserted about the intersection of the interiors of two adjacent angles? About the vertices of adjacent angles? Must adjacent angles be coplanar?

22. A polygon whose union with its interior is a convex set is a _______ polygon.

23. A polygon with ten sides is called a ___________.

24. If XY is the edge of a dihedral angle and if P and Q are points in the respective faces of the angle but not on the edge, the angle is denoted by the notation: ________. Sketch a drawing and label.

25. Use the figure below to answer Questions (a) through (e):



(a) Describe the intersection of triangle AEF and quadrilateral ABCD .

(b) Describe the intersection of segment $\overline{EF}$ and quadrilateral ABCD .

(c) Describe the union of segments $\overline{AF}$, $\overline{EF}$, and $\overline{AE}$.

(d) Describe the intersection of segments $\overline{AE}$ and $\overline{BC}$.

(e) Describe the union of triangle AEF and segment $\overline{AE}$.

Chapter 5

CONGRUENCE

5-1. Introduction.

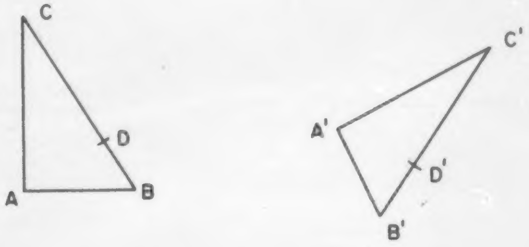
If two dresses are made from the same pattern how are they alike? If two automobile fenders are stamped by the same die, how are they alike? If two houses are made using the same set of blueprints, how are they alike?

These questions are easier to ask than they are to answer. They are concerned with the shape and size of physical objects. Much of modern economic life is based upon processes whereby many "carbon copies" of a model can be produced efficiently. The success of modern mass production technique depends upon the ability to produce physical objects which have the same size and shape.

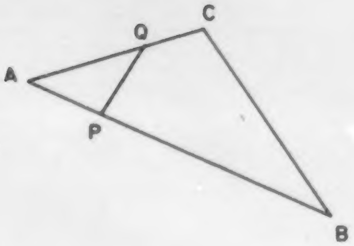
In this chapter we are concerned with the "size and shape" of geometrical objects. These geometrical objects--line segments, angles, triangles--are not physical objects; they are mathematical objects which exist in our minds. The only properties which these objects have are those given to them in the postulates and those which can be logically deduced from the postulates. The mathematical concept corresponding to "same size and shape" is "congruence." We have already discussed the concept of congruence as it applies to segments and to angles. You will remember that segments are called congruent if they have the same length, and angles are called congruent if they have the same measure.

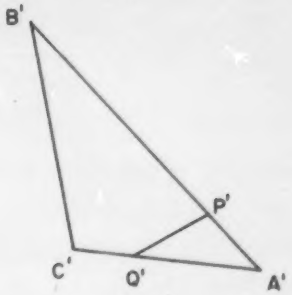
5-2. Congruence of Triangles.

Before discussing congruence of triangles (that is, triangles of the mathematical or geometrical variety) let us think briefly about the cardboard variety of triangles which we encountered in informal geometry. If two of these physical triangles have the same size and shape, it means that they can be placed one on top of the other so that they fit well together. The sides which fit together have the same lengths and the angles which fit together have the same measures. It is interesting to note that the process of fitting the triangles together establishes a one-to-one correspondence between all the points of one triangle and all the points of the other triangle. Each point of one triangle is matched with a point which lies directly above (or below) it. Each vertex is matched with a vertex, and each interior point of a side is matched with an interior point of a side. In the figures below D and D' are corresponding points.

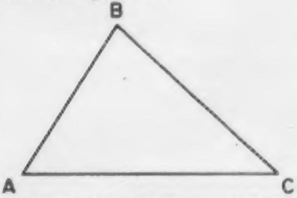


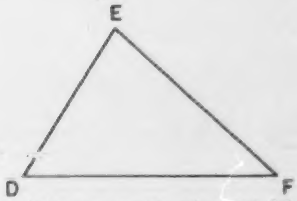
If P and Q are any two points on one of a pair of triangles having the same size and shape, and if P' and Q' are the corresponding points on the other triangles, then PQ = P'Q' .





We propose now to make the transition from the physical to the mathematical and to build a concept of congruence for geometric triangles. We do not build a mathematical theory which involves actually moving these mathematical triangles. Physical objects can be moved about; cardboard triangles can be placed in various positions to see if they fit. Mathematical points, lines, planes, triangles, etc. exist only in our minds; they have only the properties which are postulated for them or which are logical consequences of these postulates. In order to "see" if mathematical triangles "fit" we set up one-to-one correspondences between their vertices and compare the matched parts.





Given two triangles ABC and DEF there are six different one-to-one correspondences between the vertices of one and the vertices of the other:

I

$A \leftrightarrow D$
$B \leftrightarrow E$
$C \leftrightarrow F$

II

$A \leftrightarrow D$
$B \leftrightarrow F$
$C \leftrightarrow E$

III

$A \leftrightarrow E$
$B \leftrightarrow D$
$C \leftrightarrow F$

IV

$A \leftrightarrow E$
$B \leftrightarrow F$
$C \leftrightarrow D$

V

$A \leftrightarrow F$
$B \leftrightarrow D$
$C \leftrightarrow E$

VI

$A \leftrightarrow F$
$B \leftrightarrow E$
$C \leftrightarrow D$

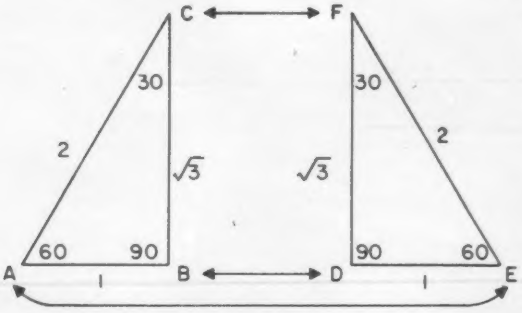
For convenience we use $ABC \longleftrightarrow DEF$ to denote the one-to-one correspondence I , $ABC \longleftrightarrow DFE$ to denote the correspondence II, and so on. Sometimes it is helpful to write $ABC \longleftrightarrow DEF$ to emphasize which vertices have been matched.

Note that there are several ways to write each correspondence. Thus $ABC \longleftrightarrow DEF$ and $CAB \longleftrightarrow FDE$ are two different ways to write the same correspondence. Each of these two correspondence symbols tells us that A and D are corresponding points, that B and E are corresponding points, and that C and F are corresponding points.

In the correspondence $ABC \longleftrightarrow DEF$, there are three pairs of corresponding points. We are interested in them primarily as points which are vertices of the angles of the triangles, and as points which are the endpoints of the sides of the triangles. In connection with this correspondence, then, we shall speak of six pairs of corresponding parts. Three of these six pairs are pairs of angles: $\angle A$ and $\angle D$, $\angle B$ and $\angle E$, $\angle C$ and $\angle F$. The other three pairs are pairs of sides: $\overline{AB}$ and $\overline{DE}$, $\overline{AC}$ and $\overline{DF}$, and $\overline{BC}$ and $\overline{EF}$.

DEFINITION. A one-to-one correspondence between the vertices of two triangles in which the corresponding parts are congruent is called a congruence between the two triangles.

Here is a picture of a correspondence between two triangles which is a congruence.

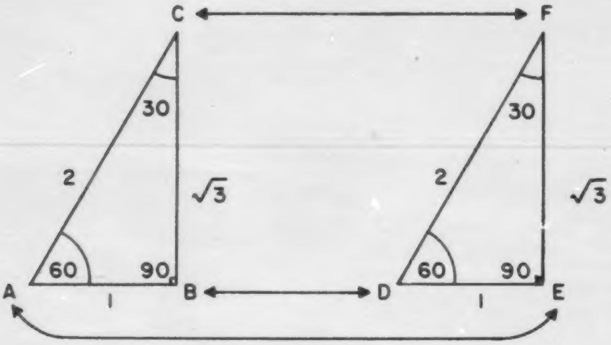


Explain why it is a congruence. Complete the following statement of a correspondence so that it correctly denotes this congruence.

ABC $\longleftrightarrow$ _____ .

Now write several other symbols which denote this same congruence.

Here is a picture of a correspondence between two triangles which is not a congruence.



ABC $\longleftrightarrow$ EDF

Explain why it is not a congruence. Complete the following statement so that it correctly denotes a congruence between the two triangles:

FED $\longleftrightarrow$ _____ .

According to our definition, a congruence is a certain kind of pairing of the vertices of triangles. It is useful to have a word to describe the existence of such a pairing.

DEFINITION. Two triangles are congruent if and only if there exists a one-to-one correspondence between their vertices which is a congruence.

If $ABC \longleftrightarrow DEF$ is a congruence between $\triangle ABC$ and $\triangle DEF$, then each of the following statements is true:

$\angle A \cong \angle D \qquad m\angle A = m\angle D$

$\angle B \cong \angle E \qquad m\angle B = m\angle E$

$\angle C \cong \angle F \qquad m\angle C = m\angle F$

$\overline{AB} \cong \overline{DE} \qquad AB = DE$

$\overline{AC} \cong \overline{DF} \qquad AC = DF$

$\overline{BC} \cong \overline{EF} \qquad BC = EF$

In each of these six lines, the congruence on the left and the equation on the right are mathematically equivalent in the sense that if one of them is true, the other is also true.

It may happen that triangles ABC and DEF are congruent even though $ABC \longleftrightarrow DEF$ is not a congruence. Triangles ABC and DEF are congruent if at least one of the one-to-one correspondences between their vertices is a congruence. To be definite we agree to use the following notation.

> Notation. $\triangle ABC \cong \triangle DEF$ means that the correspondence $ABC \longleftrightarrow DEF$ is a congruence.

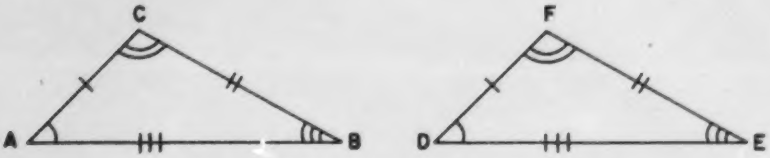
Note that if $ABC \longleftrightarrow FED$ is a congruence but $ABC \longleftrightarrow DEF$ is not a congruence, then we may write $\triangle ABC \cong \triangle FED$, but it is incorrect to write $\triangle ABC \cong \triangle DEF$.

If ABC and DEF are triangles (not necessarily distinct) and if $ABC \longleftrightarrow DEF$ is a congruence, then each of the six parts of ABC is congruent to the part of DEF with which it is matched. This is just a restatement of the definition of congruence between triangles. In view of our definition of congruent triangles, then, we sometimes say that: If triangles are congruent, their corresponding parts are congruent.

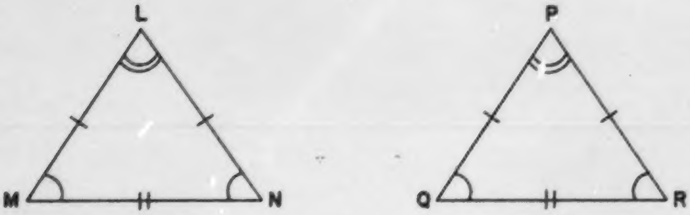
Problem Set 5-2

In Problems 1-3, two triangles are drawn and parts so marked to indicate congruence of segments and congruence of angles. For instance in Problem 1, we see $\overline{AB}$ and $\overline{DE}$ marked alike. This tells us that $\overline{AB} \cong \overline{DE}$. List all possible correspondences between the triangles in each pair and tell which are congruences.

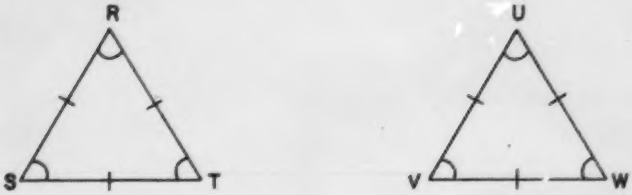
1.



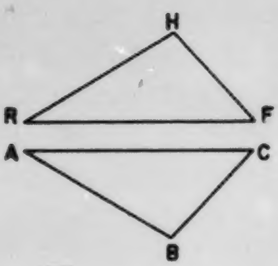
2.

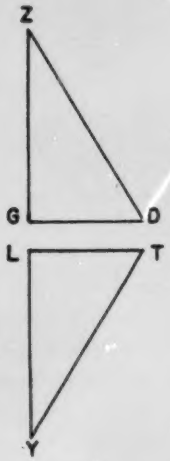


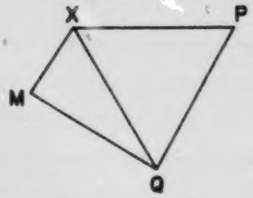
3.

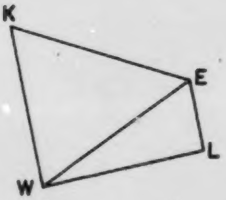


4. Below are six figures. If a correspondence looks like a congruence assume that it really is a congruence. Write the congruences between pairs of triangles. You should get four congruences in all. One is $\Delta MXQ \cong \Delta LEW$, but not $\Delta MXQ \cong \Delta LWE$.

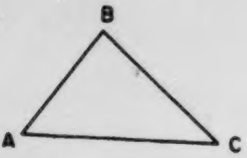


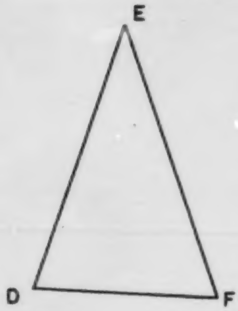


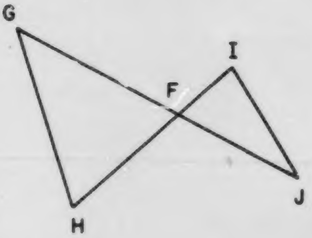


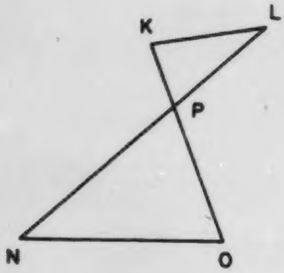


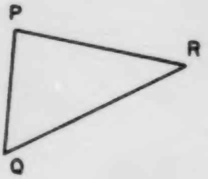
5. Answer as in Problem 4.

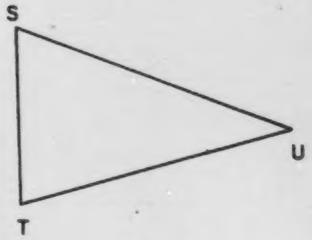




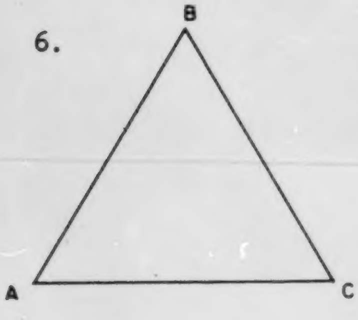








6.



In triangle ABC , the three sides are congruent and the three angles are congruent. Write down all the congruences between the triangle and itself starting with $\Delta ABC \cong \Delta ABC$. (You should get more than four congruences.)

5-3. Properties of Equality and Congruence.

We now consider the mathematical properties of a congruence. Suppose we know that $\triangle ABC \cong \triangle DEF$. We know that "$\triangle ABC$" and "$\triangle BCA$" name the same triangle. Can we therefore say that $\triangle BCA \cong \triangle DEF$? Recall that the first congruence applies to the correspondence $ABC \longleftrightarrow DEF$, while the second applies to $BCA \longleftrightarrow DEF$. If $AB \neq BC$, can both correspondences be congruences? Clearly the statements $\triangle ABC \cong \triangle LEF$ and $\triangle ABC = \triangle BCA$ have different meanings and the relation indicated by "$\cong$" is clearly different from the relation indicated by "$=$".

You may recall that the relation indicated by "$=$", called equality, was an important part of your studies in algebra. In such sentences as "$a = b$", "a" and "b" name the same object. By the substitution property of equality we may replace one of the names in a statement about an object by another of its names. Obviously, we cannot, in general, substitute one name of a triangle for another in a triangle congruence statement. For the object in such a statement is a correspondence between the vertices, and not a triangle.

Do equality and congruence relations have any properties in common? They do and they are important enough to be given names as follows:

1. The reflexive property of

equality	triangle congruence
For any a, $a = a$	For any $\triangle ABC$, $\triangle ABC \cong \triangle ABC$

To prove the reflexive property of congruence we simply note that $AB = AB$, (by the reflexive property of equality.) Therefore $\overline{AB} \cong \overline{AB}$, by the definition of congruence of segments. In similar manner we can prove $\overline{BC} \cong \overline{BC}$, $\overline{CA} \cong \overline{CA}$, $\angle A \cong \angle A$, $\angle B \cong \angle B$, $\angle C \cong \angle C$. Therefore $\triangle ABC \cong \triangle ABC$.

2. The symmetric property of

equality	triangle congruence
If $a = b$, then $b = a$	If $\triangle ABC \cong \triangle DEF$, then $\triangle DEF \cong \triangle ABC$.

Using the symmetric property of equality it is easy to prove the symmetric property of triangle congruence. This proof is left to you as a problem in the next problem set.

3. The transitive property of

equality	triangle congruence
If $a = b$ and $b = c$, then $a = c$.	If $\triangle ABC \cong \triangle DEF$ and $\triangle DEF \cong \triangle GHK$, then $\triangle ABC \cong \triangle GHK$.

Using the transitive property of equality we indicate how to prove the transitive property of triangle congruence. From $\triangle ABC \cong \triangle DEF$, it follows that $\overline{AB} \cong \overline{DE}$. Because congruent segments have equal lengths, $AB = DE$. In the same manner we may start with $\triangle DEF \cong \triangle GHK$ and conclude, $DE = GH$. By the transitive property of equality we may then say, $AB = GH$, and therefore $\overline{AB} \cong \overline{GH}$. In this manner we prove that, in the correspondence $ABC \longleftrightarrow GHK$, all pairs of corresponding sides are congruent. Now we consider corresponding angles. Because $\triangle ABC \cong \triangle DEF$, it follows that $\angle A \cong \angle D$ and $m\angle A = m\angle D$; from $\triangle DEF \cong \triangle GHK$, we can conclude, $m\angle D = m\angle G$. Therefore $m\angle A = m\angle G$ or $\angle A \cong \angle G$. Similarly all pairs of corresponding angles of $\triangle ABC$ and $\triangle GHK$ are congruent and $\triangle ABC \cong \triangle GHK$.

Our discussions of congruences in the preceding chapters and in the above paragraphs can be summarized in three theorems.

THEOREM 5-1. Congruence for segments has the following properties.

Reflexive: $\overline{AB} \cong \overline{AB}$.

Symmetric: If $\overline{AB} \cong \overline{CD}$ then $\overline{CD} \cong \overline{AB}$.

Transitive: If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$.

THEOREM 5-2. Congruence for angles has the following properties.

Reflexive: $\angle A \cong \angle A$.

Symmetric: If $\angle A \cong \angle B$, then $\angle B \cong \angle A$.

Transitive: If $\angle A \cong \angle B$, and $\angle B \cong \angle C$, then $\angle A \cong \angle C$.

THEOREM 5-3. Congruence for triangles has the following properties.

Reflexive: $\triangle ABC \cong \triangle ABC$.

Symmetric: If $\triangle ABC \cong \triangle DEF$, then $\triangle DEF \cong \triangle ABC$.

Transitive: If $\triangle ABC \cong \triangle DEF$ and $\triangle DEF \cong \triangle GHI$, then $\triangle ABC \cong \triangle GHI$.

We note next that equality has two properties which congruence does not have. They are the addition and multiplication properties listed below.

The addition property of equality.

If a , b and x are numbers and $a = b$, then $a + x = b + x$; more generally, if a , b , c , d are numbers, $a = b$, and $c = d$, then $a + c = b + d$.

Let us note that $-x$ also represents a number and then $a + (-x)$ denotes the difference $a - x$. Therefore, the addition property applies to subtraction also.

Illustration 1.

Suppose that A , B , C , D are collinear points in that order and that $AB = CD$. Then the addition property of equality permits the conclusion, $AB + BC = CD + BC$. Then $AC = BD$ follows by the substitution property of equality. For $AC = AB + BC$ and $BD = CD + BC$.

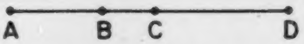
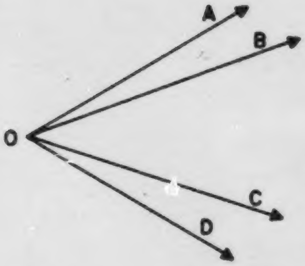


Illustration 2.

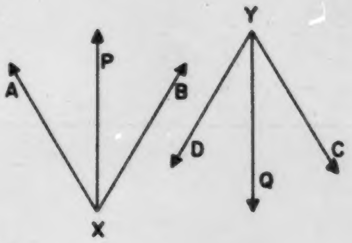


Suppose $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ are concurrent rays in that order, and that $m \angle AOC = m \angle BOD$. Then the addition property of equality tells us that $m \angle AOC - m \angle BOC = m \angle BOD - m \angle BOC$. Explain why $m \angle AOB = m \angle COD$ follows from the substitution property of equality.

The multiplication property of equality.

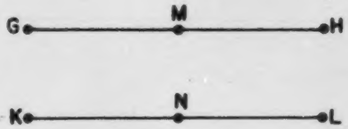
If a , b and x are numbers and $a = b$, then $ax = bx$; more generally, if a , b , c , d are numbers, $a = b$, and $c = d$, then $ac = bd$.

Illustration 1.



Suppose $\overrightarrow{XP}$ is the midray of $\angle AXB$, $\overrightarrow{YQ}$ is the midray of $\angle CYD$ and $m \angle AXB = m \angle CYD$. Then the multiplication property of equality, with the multiplier $\frac{1}{2}$, permits the conclusion $m \angle AXP = m \angle CYQ$ or $m \angle AXP = m \angle DYQ$.

Illustration 2.



If M is the midpoint of $\overline{GH}$ and N is the midpoint of $\overline{KL}$ and $MH = NL$ then the multiplication property of equality, with the multiplier 2, tells us that $GH = KL$.

Problem Set 5-3a

1. State the property which justifies each of the following statements.

   (a) $\overline{AB} \cong \overline{AB}$.

   (b) $AB = AB$.

   (c) If $\angle ABC \cong \angle DEF$, then $\angle DEF \cong \angle ABC$.

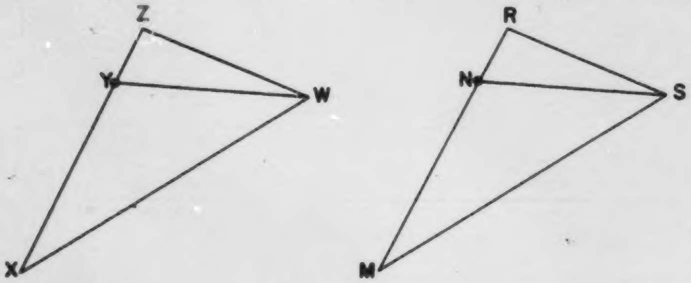
   (d) If $\triangle PQR \cong \triangle RST$ and $\triangle RST \cong \triangle XYZ$, then $\triangle PQR \cong \triangle XYZ$.

   (e) If $a = b$ and $b = c$, then $a = c$.

   (f) If $2x + 5 = 12$, then $2x = 7$.

   (g) If $3x = 15$, then $x = 5$.

In each of the six problems below (2 through 7) there appears a list of statements referring to the diagrams. You are expected to justify each statement. Having justified a statement, you may then use it in helping to justify later statements (if any) in the same problem. As indicated in the drawings Y is in $\overline{XZ}$, N is in $\overline{MR}$, $\overrightarrow{WY}$ is between $\overrightarrow{WX}$ and $\overrightarrow{WZ}$, $\overrightarrow{SN}$ is between $\overrightarrow{SM}$ and $\overrightarrow{SR}$.



2. (a) $XY + YZ = XZ$ and $MN + NR = MR$.

   (b) If $XY = MN$ and $YZ = NR$, then $XY + YZ = MN + NR$.

   (c) $XZ = MR$.

3. (a) $m\angle ZWX - m\angle ZWY = m\angle XWY$ and $m\angle RSM - m\angle RSN = m\angle MSN$.

   (b) If $m\angle ZWX = m\angle RSM$ and $m\angle ZWY = m\angle RSN$, then $m\angle ZWX - m\angle ZWY = m\angle RSM - m\angle RSN$.

   (c) $m\angle XWY = m\angle MSN$.

4. (a) If Y is the midpoint of $\overline{XZ}$ and N is the midpoint of $\overline{MR}$, then $\frac{1}{2}XZ = XY$ and $\frac{1}{2}MR = MN$.

(b) If $XZ = MR$, then $\frac{1}{2}XZ = \frac{1}{2}MR$.

(c) $XY = MN$.

5. (a) If $\overrightarrow{WY}$ is the midray of $\angle ZWX$ and $\overrightarrow{SN}$ is the midray of $\angle RSM$, then $2 \cdot m\angle ZWX = m\angle ZWX$ and $2 \cdot m\angle RSN = m\angle RSM$.

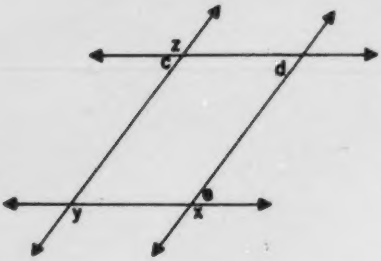
(b) If $m\angle ZWY = m\angle RSN$, then $2 \cdot m\angle ZWY = 2 \cdot m\angle RSN$.

(c) $m\angle ZWX = m\angle RSM$.

6. If $XY = MN$ and $MN = NR$, then $XY = NR$.

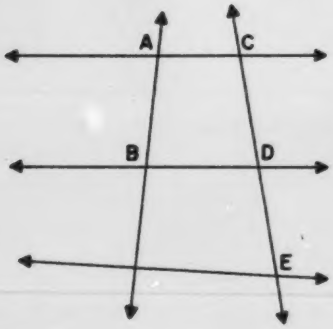
7. If $m\angle ZWY = m\angle YWX$ and $m\angle ZWY = m\angle RSN$, then $m\angle YWX = m\angle RSN$.

8. Which property of congruence, equality or betweenness justifies each of the following statements? Consult adjacent diagrams before answering.



(a) If $\angle c \cong \angle d$, $\angle d \cong \angle e$, then $\angle c \cong \angle e$.

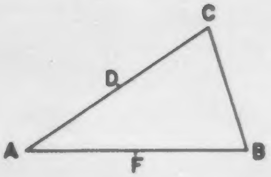
(b) If $m\angle z = m\angle y = m\angle x$, then $m\angle z = m\angle x$.



(c) If $\overline{AB} \cong \overline{CD}$, $\overline{CD} \cong \overline{DE}$, then $\overline{AB} \cong \overline{DE}$.

(d) If $AC = AB$ and $DE = AB$, then $AC = DE$.

9.

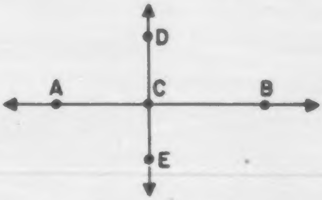


Suppose $AC = BA$; D , F are midpoints of $\overline{AC}$, $\overline{BA}$, respectively.

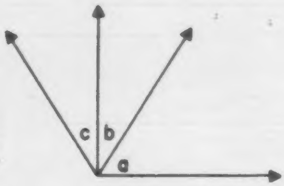
Give an argument to show that $\overline{DC} \cong \overline{FB}$.

10. Which of the following conclusions follow from the hypotheses? Give reasons.

(a) If $\overleftrightarrow{AB} \perp \overleftrightarrow{DE}$ and $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at C , then $\overleftrightarrow{AC} \perp \overleftrightarrow{DE}$.



(b)



Hypothesis: $m\angle a + m\angle b = 90$
$\angle b \cong \angle c$

Conclusion: $\angle c$ is the complement of $\angle a$.

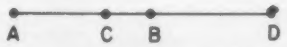
*11. Prove the symmetric property of triangle congruence.

Suppose that A , B , C , D are collinear points in that order and that $\overline{AB} \cong \overline{CD}$, we can then argue that $\overline{AC} \cong \overline{BD}$ as follows:

1. $AB = CD$, because, if segments are congruent, then they have the same length.
2. $BC = BC$, because of the reflexive property of equality.
3. $AB + BC = CD + BC$, because of the addition property of equality.
4. $AB + BC = AC$, and $CD + BC = BD$, because of the Betweenness Distance Theorem.
5. $AC = BD$, because of the substitution property of equality.

6. $\overline{AC} \cong \overline{BD}$, because, if segments have the same length then they are congruent.

From the statement $\overline{AB} \cong \overline{CD}$ and the collinearity of A , B , C , D in that order we are thus able to deduce that $\overline{AC} \cong \overline{AD}$.



Suppose that the order of points were A , C , B , D , and that $\overline{AB} \cong \overline{CD}$. Check the above argument step by step, for this new hypothesis. If Step 3 were $AB - BC = CD - BC$ and Step 4 were $AB - BC = AC$, $CD - BC = BD$, would the argument then be valid?

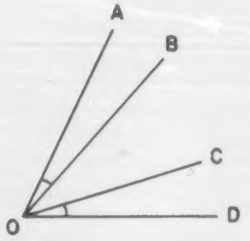
We can combine both hypotheses in the single statement: B and C are between A and D . Then we can state the following theorem.

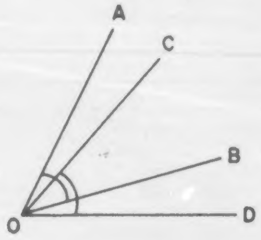
THEOREM 5-4. (Betweenness-Addition Theorem for Points) If points B and C are between A and D and $\overline{AB} \cong \overline{CD}$, then $\overline{AC} \cong \overline{BD}$.

By a similar argument we can prove a corresponding theorem for rays.

THEOREM 5-5. (Betweenness-Addition Theorem for Rays) If $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are between $\overrightarrow{OA}$ and $\overrightarrow{OD}$ and $\angle AOB \cong \angle COD$, then $\angle AOC \cong \angle BOD$.

This theorem can be proved by considering two cases.

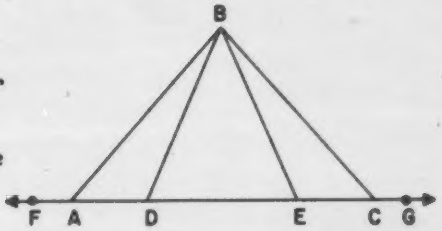




The proof is left for you as a problem in the next problem set.

Problem Set 5-3b

1. State which theorem may be used to justify each of the following statements. Refer to the adjacent diagram in which the points on $\overleftrightarrow{FG}$ are collinear in the order in which they appear.



   (a) If $\overline{AD} \cong \overline{EC}$, then $\overline{AE} \cong \overline{DC}$.
   (b) If $\angle ABE \cong \angle CBD$, then $\angle ABD \cong \angle CBE$.
   (c) If $\overline{FC} \cong \overline{AG}$, then $\overline{FA} \cong \overline{CG}$.
   (d) If $\angle ABD \cong \angle CBE$, then $\angle ABE \cong \angle CBD$.

*2. Prove Theorem 5-5.

5-4. Definitions in Proofs.

The proof of Theorem 5-4 uses the definition of congruent segments twice. First we use it to conclude that if two segments are congruent, then they have the same length. Later in the proof we use it to conclude that if two segments have the same length, then they are congruent. Does the definition work both ways?

We agreed to call segments having the same measure congruent segments. This is actually an agreement that the phrases "segments having the same length " and "congruent segments" are interchangeable names of the same thing. Therefore we may write the definition in two parts as follows:

(1) If segments are congruent, then they have the same length.
(2) If segments have the same length, then they are congruent.

Let us call this form of a definition, the complete form. Sometimes we abbreviate these two statements as follows:

> Segments are congruent if and only if they have the same length.

Let us call this form of a definition the "if and only if" form. We note two important things about the complete form of a definition:

(a) It has two parts, written in the "if-then" form. The if-clause of each statement is the then-clause of the other. Of course this means that the then-clause of each is the if-clause of the other.

(b) One part of a definition may be used to support one statement while the other part may be used to support another statement. For an example, see Steps 1 and 6 in the proof of Theorem 5-4. Of course, we must always be careful to use the appropriate part.

Problem Set 5-4

1. One of the two parts of the complete form of a definition is given in each statement below. Write the other part for each.
   (a) If an angle is a right angle, then its measure is 90 .
   (b) If a plane contains all the points of a set, then those points are coplanar.
   (c) If two rays have a common endpoint and do not lie on the same line, then their union is an angle.
   (d) If $\overrightarrow{VP}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VB}$ and $\angle AVP \cong \angle PVB$, then $\overrightarrow{VP}$ is the midray of $\angle AVB$.
2. For each of the following definitions write the two statements in the complete form of a definition.
   (a) The point whose coordinate is 1 in a coordinate system is called the unit point of that coordinate system.
   (b) Two distinct collinear rays with a common endpoint are called opposite rays.
   (c) The midpoint of a segment is the point which belongs to the segment and is equally distant from the endpoints of the segment.

3. Using the "if and only if" form, write the definition of

   (a) obtuse angle.
   (b) vertical angles.
   (c) linear pair.

4. In each of the following tell whether or not the part of the definition used as a reason can possibly justify the given statement. If it cannot, explain why.

   (a) $AB = CD$, because if segments are congruent they have the same length.
   (b) $AB = CD$, because if segments have the same length, they are congruent.
   (c) $\overrightarrow{AB}$ is a midray, because, if a ray is the midray of angle it bisects the angle.
   (d) $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$, because if two lines are perpendicular they form right angles.
   (e) $\angle ABC \cong \angle DBE$, because if two angles are vertical, they are congruent.

5. Using one part of the complete form of a definition tell why each of the following statements is true. If you need to, draw a diagram.

   (a) If $\overleftrightarrow{RS} \perp \overleftrightarrow{RT}$, then $\angle SRT$ is a right angle.
   (b) If M is the midpoint of $\overline{AB}$, then $AM = MB$.
   (c) If $m\angle A + m\angle B = 180$, then $\angle A$ and $\angle B$ are supplementary.
   (d) If $\angle A$ and $\angle B$ are complementary, then $m\angle A + m\angle B = 90$.
   (e) If S is a convex set of points and P and Q are in the set, then $\overline{PQ}$ is in S .
   (f) If $\Delta ABC \cong \Delta DEF$, then $\overline{AB} \cong \overline{DE}$.
   (g) If P is an interior point of $\angle BAC$, then $\overrightarrow{AP}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
   (h) If $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then D is an interior point of $\angle BAC$.

5-5. Proofs in Two-Column Form.

As you know, many of our postulates and theorems are conditionals, that is, they have the form "if p then q" , where p and q are statements. In the previous section we saw that the complete form of a definition has two parts each of which also has the if-then form. In order to understand mathematical proof better we examine how such statements are used in proofs.

Let us consider the following statement concerning two persons named X and Y , about whom we know nothing.

A. If X is Y's father's brother, then X is Y's uncle.

We see that A involves the following two statements.

B. X is Y's father's brother. [This part of A is called the hypothesis.]

C. X is Y's uncle. [This part of A is called the conclusion.]

Even though we don't know X and Y , can we say anything about the truth of statements A, B, and C ? Do we know if B is true? Do we know if C is true? How about Statement A ? We can easily see that even though B need not be true, and that C need not be true, that A is true. Thus a conditional may be true even though its hypothesis and conclusions are not.

Suppose, now, that A and B are both true. Then it follows logically that C is true. Check with our example. This is a most important concept in mathematical proofs. It means that we can assert C after we have proved or know that both A and B have been established. But it does not mean that B follows from A and C . In general:

> If a conditional and its hypothesis have been established, then its conclusion is established.

More concisely, if we know that the following two statements are established

(1) if "B" then "C"
(2) "B" ,

then we may conclude that "C" is established.

A two-column proof of a theorem is a chain of statements written in one column, supported by reasons written in another column. The proof shows how the conclusion of a theorem follows logically from its hypothesis. When a statement in this chain is established because it is part of the hypothesis of the theorem, we simply write "hypothesis" as the reason. Otherwise, a statement may have as its supporting reason a combination of a conditional and its hypothesis. Our example shows this when we say that the combination of A and B is a reason for saying C . The conditional is acceptable if it is a postulate, or part of a definition, or an already proved theorem. Furthermore, the hypothesis of this conditional should have appeared as an earlier statement in the proof. Finally, the conclusion of the conditional should "fit" the statement that is being supported. We can summarize these relations in a diagram as follows:

Statement	Reason
P	-------
Q	If P then Q , and P .

We now show an example of such a proof. Note that in those reasons which are conditionals, we write the numbers of the statements in which we have established the hypothesis of the conditional.

We use the theorem regarding the transitive property of congruence for segments as our example:

THEOREM: If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$.

Proof:

Hypothesis: $\overline{AB} \cong \overline{CD}$, $\overline{CD} \cong \overline{EF}$.
Conclusion: $\overline{AB} \cong \overline{EF}$.

	Statement		Reason
1.	$\overline{AB} \cong \overline{CD}$.	1.	Hypothesis.
2.	$\overline{CD} \cong \overline{EF}$.	2.	Hypothesis.

(We might have written Statements 1 and 2 on one line because they have the same reason.)

3.	$AB = CD$, $CD = EF$.	3.	If line segments are congruent (1,2), then they have the same measure.
4.	$AB = EF$.	4.	If $a = b$ and $b = c$ (3), then $a = c$.
5.	$\overline{AB} \cong \overline{EF}$.	5.	If line segments have the same measure (4), then they are congruent.

Let us note several important points about this proof.

(1) The proof is completed when the last statement in the first column is the same as the conclusion.

(2) When a statement is part of the hypothesis we write "hypothesis" as its reason.

(3) When a reason is in the if-then form, its hypothesis refers to an earlier statement or statements for support. For instance the if-clause of Reason 3 refers to Statements 1 and 2. But the then-clause of Reason 3 refers to Statement 3.

(4) When a reason is not in the if-then form, and it can be written in that form, then it must satisfy the requirements stated in (3) above.

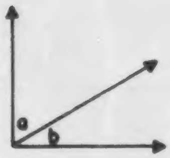
Problem Set 5-5

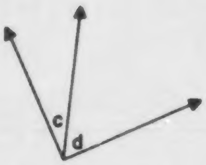
1. Write each of the following theorems in "if-then" form. State the hypothesis and conclusion of each.
   - (a) Vertical angles are congruent.
   - (b) Right angles are congruent.
   - (c) Complements of congruent angles are congruent.
   - (d) A line and a point not in the line are contained in exactly one plane.
   - (e) The interior of an angle is a convex set.
   - (f) The intersection of any two convex sets of points is a convex set.

2. In the following example of a two-column proof several reasons are given in the "if-then" form. In each such reason, tell to which statement the if-clause refers and to which statement the then-clause refers.

THEOREM: Complements of congruent angles are congruent.

(Look at your "if-then" statement for Problem 1 (c) above.)





Hypothesis: $\angle a$ is the complement of $\angle b$.
$\angle d$ is the complement of $\angle c$.
$\angle b \cong \angle c$

The conclusion we are to prove is $\angle a \cong \angle d$.

Statement	Reason
1. $\angle a$ is the complement of $\angle b$, $\angle d$ is the complement of $\angle c$.	1. Hypothesis.
2. $m \angle a + m \angle b = 90$ $m \angle d + m \angle c = 90$.	2. If two angles are complementary then the sum of their measures is 90 . (Definition of complementary angles.)
3. $m \angle a = 90 - m \angle b$. $m \angle d = 90 - m \angle c$.	3. If x, y, z are numbers and $x = y$, then $x - z = y - z$. (The addition property of equality.)
4. $\angle b \cong \angle c$.	4. Hypothesis.
5. $m \angle b = m \angle c$.	5. If two angles are congruent then they have the same measure. (Definition of congruence for angles.)
6. $90 - m \angle b = 90 - m \angle c$.	6. If a, b, x, y are numbers and if $a = b$, $x = y$, then $a - x = b - y$. (The addition property of equality.)
7. $m \angle a = m \angle d$.	7. If $x = y = z = w$ then $x = w$. (Transitive property of equality.)
8. $\angle a \cong \angle d$.	8. If two angles have the same measure, then they are congruent.

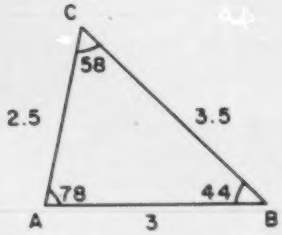
3. Rewrite in two-column form the paragraph proofs given for the following theorems in Chapter 4.
   (a) Right angles are congruent.
   (b) Supplements of congruent angles are congruent.
   (c) Vertical angles are congruent.

4. Identify Reasons 3, 4, and 5 in the example of Section 5-5 either as a postulate, definition, theorem, or property.

5-6. Some Experiments and Some Postulates.

In order to understand triangle congruences better we shall perform experiments with physical objects. We shall prepare to discuss congruent triangles by making measurements of parts of cardboard triangles. In a preceding section we defined two mathematical triangles as being congruent if there is a congruence between them--if there is a matching of sides and angles so that the matched parts are congruent. At this point in our study we need to establish six congruences involving sides and angles before we can be sure that two triangles are congruent. What is the corresponding situation for cardboard triangles? In order to know that two such triangles will fit well together, do we need to measure all three sides and all three angles?

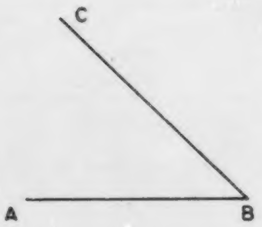
Experiment 1. In the SMSG workroom there is a cardboard triangle ABC which we have measured carefully. The inch-measures of its sides and the degree-measures of its angles are as follows: $m\angle A = 78$, $m\angle B = 44$, $m\angle C = 58$, $AB = 3$, $BC = 3.5$, $CA = 2.5$. Below is a sketch of triangle ABC.

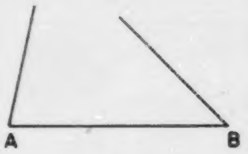


Using a pencil, ruler, protractor, compass, scissors and cardboard (all of them or as many as you wish) you should make a triangle which you think will be an accurate copy of the workroom triangle ABC . Some of you may start by drawing $\overline{AB}$. Others may start with $\angle A$. After you have finished making your triangle, review what you have done and make a list of the steps in the order in which you did them.

Perhaps you have one of these lists:

List I	List II	List III
1. Draw $\overline{AB}$.	1. Draw $\angle A$.	1. Draw $\overline{AB}$.
2. Draw $\angle B$.	2. Draw $\overline{AB}$.	2. Draw arc from A as center and AC as radius.
3. Draw $\overline{BC}$.	3. Draw $\angle B$.	3. Draw the arc from B as center and BC as radius.
4. Close the triangle by connecting A and C with $\overline{AC}$.	4. Close the triangle by drawing the sides of $\angle A$ and $\angle B$ long enough.	4. Close the triangle by connecting the point of intersection of the arcs to A and to B .





The figures in each list show the stage of the drawing before closing the triangle.

Look at each list. How many measures (either of side or angle) are used in the first list? In the second? In the third? Were all six measures needed to duplicate the triangle?

Let us look at the first list again. How many side measures are used? How many angle measures? Is the angle between the two sides? This combination of two sides and the angle between them is called two sides and the included angle. It is abbreviated by the symbol S.A.S. Name another S.A.S. combination in triangle ABC .

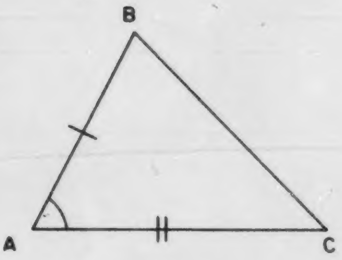
Let us now look at the second list again. What combination of measures is used? The side mentioned is said to be included between the two angles mentioned. This combination is referred to by the symbol A.S.A. Name another A.S.A. combination.

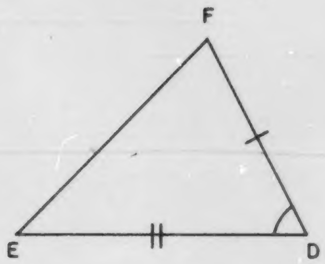
Look at the third list. We can readily see that it mentions three sides. This combination is referred to by the symbol S.S.S.

Experiment 2. Construct a cardboard triangle ABC so that $AB = 4$, $BC = 3$, $m\angle B = 35$. Then measure $\overline{CA}$, $\angle A$, $\angle C$. Compare your results with a classmate. Make allowance for inaccuracy in your measurements.

What conclusion can we draw? Do you agree that it suggests the following postulate for mathematical triangles?

> Postulate 19. (The S.A.S. Postulate) Given a one-to-one correspondence between the vertices of two triangles (not necessarily distinct). If two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.





For the triangles suggested by the pictures this postulate means that if

$$\overline{AB} \cong \overline{DF}$$
$$\angle A \cong \angle D$$
$$\overline{AC} \cong \overline{DE} \ ,$$

then

$$\triangle BAC \cong \triangle FDE \ ;$$

that is,

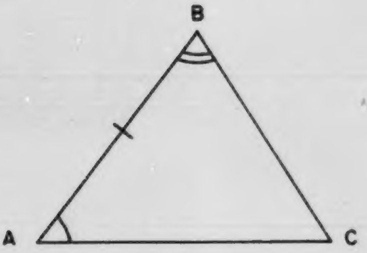
$$BAC \longleftrightarrow FDE$$

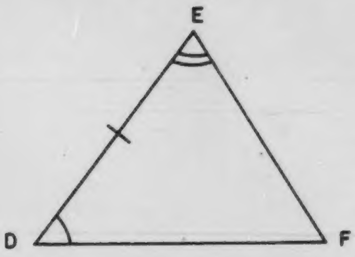
is a congruence.

Experiment 3. Construct a cardboard triangle ABC so that $m\angle A = 40$, $AB = 3.75$, $m\angle B = 48$. Then measure $\overline{BG}$, $\overline{AC}$, $\angle C$. Compare these results with a classmate.

What conclusion can we draw? Do you agree that it suggests the following postulate for mathematical triangles?

> Postulate 20. (The A.S.A. Postulate) Given a one-to-one correspondence between the vertices of two triangles (not necessarily distinct). If two angles and the included side of one triangle are congruent to the corresponding parts of the other triangle, then the correspondence is a congruence.

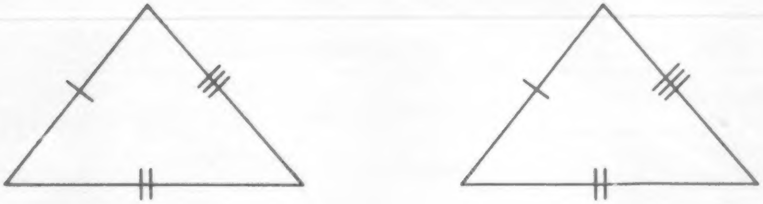




Experiment 4. Construct a cardboard triangle ABC so that AB = 5 , BC = 4 , AC = 3.5 . Then measure $\angle A$, $\angle B$, $\angle C$. Compare results with a classmate.

What conclusion can we draw? Do you agree that it suggests the following postulate for mathematical triangles?

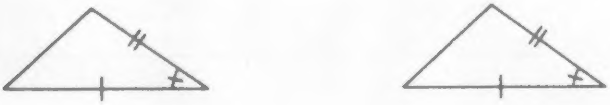
Postulate 21. (The S.S.S. Postulate) Given a one-to-one correspondence between the vertices of two triangles (not necessarily distinct). If three sides of one triangle are congruent to the corresponding sides of the other triangle, then the correspondence is a congruence.



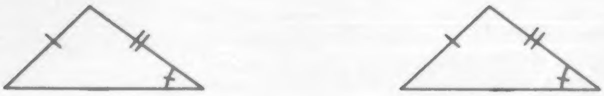
Problem Set 5-6

1. In each pair of triangles like markings indicate congruent parts. Which pairs of triangles could be proved congruent? In each case of congruence state the postulate you would use (A.S.A., S.A.S., S.S.S.).

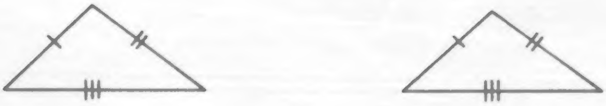
(a)



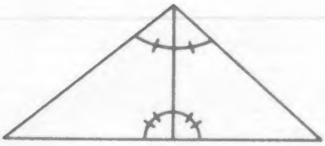
(b)



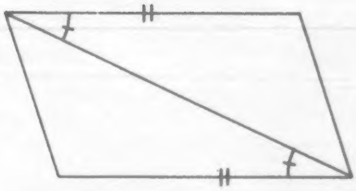
(c)



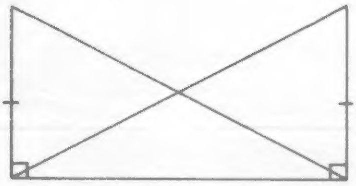
(d)



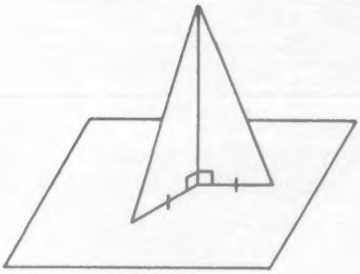
(e)



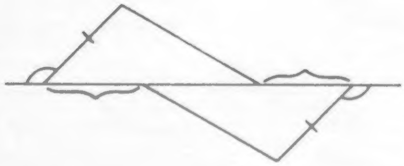
(f)



(g)



(h)



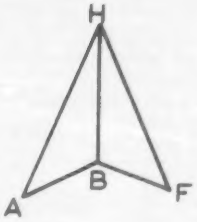
2. In accordance with the combination indicated fill the blanks with the correct symbols.

   (a) Side, angle, side of $\triangle ABH$:
   $\overline{AH}$, _____ , $\overline{HB}$.

   (b) Angle, side, angle of $\triangle ABH$:
   _____ , $\overline{HB}$, _____ .

   (c) Angle, side, angle of $\triangle BFH$:
   $\angle F$, _____ , $\angle HBF$.

   (d) Side, angle, side of $\triangle BFH$:
   $\overline{BF}$, _____ , _____ .



3. Follow the directions of Problem 2.

   (a) Angle, side, angle of $\triangle ABF$:
   _____ , $\overline{BF}$, _____ .

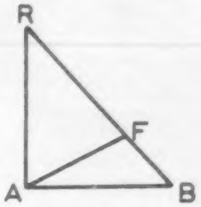
   (b) Side, angle, side of $\triangle RAF$:
   _____ , $\angle R$, _____ .

   (c) Side, angle, side of $\triangle RAB$:
   _____ , $\angle B$, _____ .

   (d) Side, angle, side of $\triangle RAB$:
   $\overline{BR}$, _____ , $\overline{RA}$.

   (e) Angle, side, angle of $\triangle RAF$:
   $\angle R$, _____ , $\angle RFA$.

   (f) Angle, side, angle of $\triangle AFB$:
   $\angle FAB$, $\overline{AF}$, _____ .



4. Follow the directions of Problem 2.

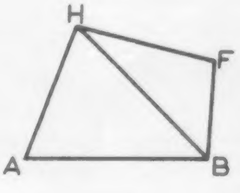
   (a) Side, angle, side of $\triangle HFB$:
   _____ , $\angle HBF$, _____ .

   (b) Angle, side, angle of $\triangle ABH$:
   _____ , $\overline{HB}$, _____ .

   (c) Side, angle, side of $\triangle HFB$:
   $\overline{HB}$, _____ , $\overline{BF}$.

   (d) Angle, side, angle of $\triangle HFB$:
   _____ , $\overline{BF}$, _____ .

   (e) Side, angle, side of $\triangle ABH$:
   $\overline{AH}$, _____ , $\overline{AB}$.

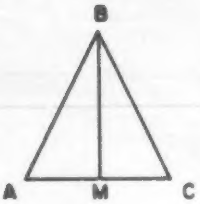


5. In each of the following problems two triangles appear to be congruent. Assume that all points in each diagram are coplanar, that points which appear to be on lines are on those lines, and that the rays and points are in the relative positions suggested by the diagram.

   (i) Mark each diagram, as was done in Problem 1, to show the given information.

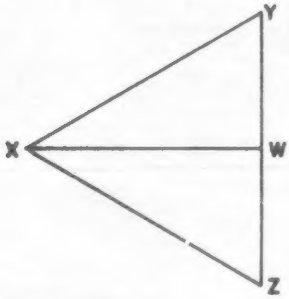
   (ii) If the given information is sufficient to prove the triangles congruent, state the postulate you would use (S.A.S., A.S.A., S.S.S.). If the information given is insufficient, write "Insufficient."

(a)



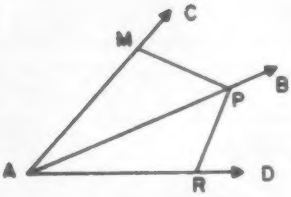
M is the midpoint of $\overline{AC}$.
$\overline{AB} \cong \overline{CB}$.

(b)



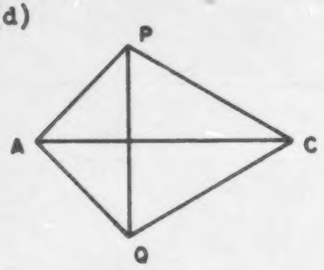
$\overline{XW} \perp \overline{YZ}$ at W .

(c)



$\overrightarrow{AB}$ is the midray of $\angle CAD$.
$m\angle MPA = m\angle RPA$.

(d)



Consider only $\triangle APC$ and $\triangle AQC$.

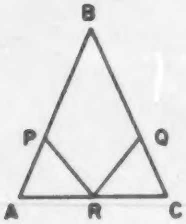
$\overline{AP} \cong \overline{AQ}$,

$\overline{PC} \cong \overline{QC}$,

$m\angle APQ = m\angle AQP$,

$m\angle CPQ = m\angle CQP$.

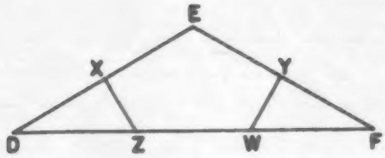
(e)



Point P lies between A and B. Point Q lies between C and B.

$\overline{AP} \cong \overline{CQ}$; R is the midpoint of $\overline{AC}$; $\overline{RP} \cong \overline{RQ}$.

(f)



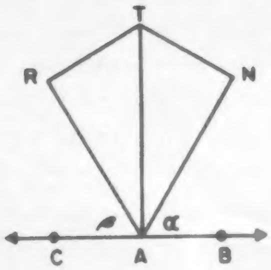
$\overline{DE} \cong \overline{EF}$.

X is midpoint of $\overline{DE}$.

Y is midpoint of $\overline{EF}$.

$\angle DXZ$ and $\angle FYW$ are right angles.

(g)

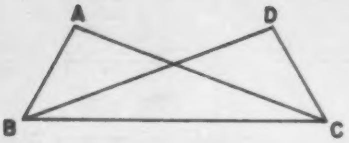


Assume: $\overline{AT} \perp \overline{CB}$,

$\angle\alpha \cong \angle\beta$,

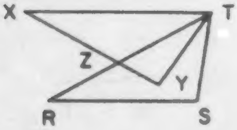
$\overline{AN} \cong \overline{AR}$.

(h)



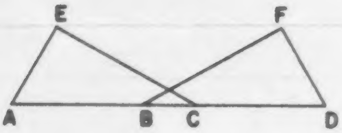
$\angle ABD \cong \angle DCA$
$\angle DEC \cong \angle ACB$

(i)



$\overline{TY} \cong \overline{TS}$,
$\overline{RT} \cong \overline{XT}$,
$\angle XTZ \cong \angle STY$.

(j)



$\overline{AB} \cong \overline{CD}$
$\angle A \cong \angle D$
$\overline{AE} \cong \overline{FD}$

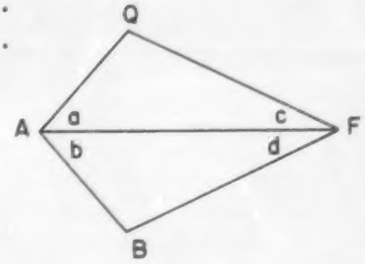
6. If a figure appears beside the statement of a problem, you are to assume that points which appear to be collinear in a certain order are collinear in that order, and that rays which appear to be concurrent in a certain order are concurrent in that order.

In some parts of this exercise there is not enough information to enable you to prove the two triangles are congruent even if you use all other facts that you know, for example, that "vertical angles are congruent." If it can be proved that the two triangles are congruent, name the statement (A.S.A. or S.A.S.) supporting your conclusion; if there is not enough information given to prove the triangles are congruent, name another pair of parts which if they were congruent would enable you to prove the triangles congruent. If there are two possibilities, name both.

(a) Given only that $\overline{AH} \cong \overline{AB}$.

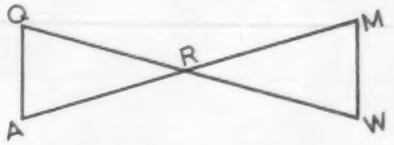
(b) Given only that $\angle c \cong \angle d$.

(c) Given only that $\angle a \cong \angle b$ and $\angle c \cong \angle d$.



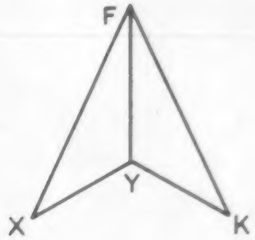
(d) Given only that $\overline{AR} \cong \overline{MR}$.

(e) Given only that $\angle A \cong \angle M$.



(f) Given only that $\angle XFY \cong \angle KFY$.

(g) Given only that $\angle XYF \cong \angle KYF$.



5-7. Writing Proofs Involving Triangle Congruences.

From this point on writing your own proofs will be a major project. We show some examples which will help you to find and write proofs. They will involve congruences between triangles, between segments and between angles.

Example 1. If $\angle AFB$ and $\angle RFH$ are a pair of vertical angles, $\overline{AF} \cong \overline{FR}$ and $\overline{BF} \cong \overline{RH}$, then $\overline{AB} \cong \overline{RH}$.

In starting to construct a proof for this statement we first examine it to see what the hypothesis is and then we draw a figure that seems to fit this hypothesis.

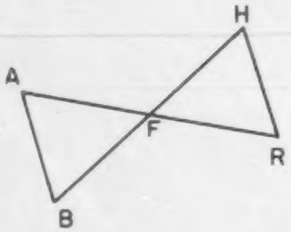


Figure (1)

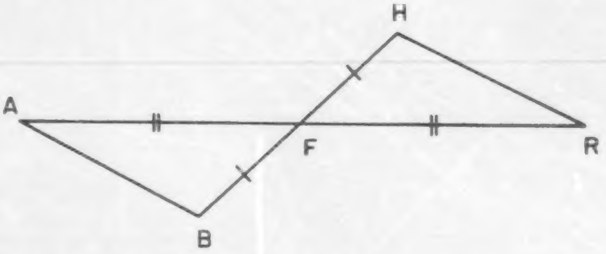
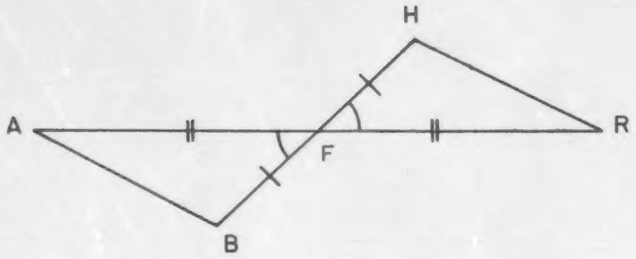


Figure (2)

Figure (1) and Figure (2) both fit. Why might Figure (2) be better? We mark $\overline{AF}$ and $\overline{RF}$ alike to show that the hypothesis tells us that they are congruent. We also mark $\overline{BF}$ and $\overline{HF}$ alike for the same reason. Now we look for a plan for the proof. We can show that $\overline{AB} \cong \overline{RH}$, as required in the conclusion, if we can show that they are corresponding parts of congruent triangles. Which triangles? We are told that $\angle AFB$ and $\angle RFH$ are vertical angles, and hence they are congruent. We now have enough evidence to use the S.A.S. Postulate, if we use the correspondence $ABF \longleftrightarrow RHF$. We can now write the proof.

Hypothesis: $\angle AFB$ and $\angle RFH$ are vertical angles.
$\overline{AF} \cong \overline{RF}$
$\overline{BF} \cong \overline{HF}$



Conclusion to be proved: $\overline{AB} \cong \overline{RH}$

Statements	Reasons
1. $\overline{AF} \cong \overline{RF}$. $\overline{BF} \cong \overline{HF}$.	1. Hypothesis
2. $\angle AFB$ and $\angle RFH$ are vertical angles.	2. Hypothesis
3. $\angle AFB \cong \angle RFH$.	3. If angles are vertical (2), then they are congruent.
4. $\triangle ABF \cong \triangle RHF$.	4. S.A.S. Postulate, (1,3).
5. $\overline{BF} \cong \overline{HF}$.	5. If triangles are congruent (4), then corresponding parts are congruent.

You should note that in the reason for Statement 3, that the if-clause ends with "(2)". This means that the hypothesis of the conditional was established in Statement 2.

The reason for Statement 4 is the S.A.S. Postulate. We use the name for the postulate because it is a long statement. Note that it is a conditional. The symbol "(1,3)" refers to Statements 1 and 3 which supply the hypothesis of the conditional. The conclusion of the conditional gives us Statement 4.

Verify that (4) is the correct reference to support Reason 5.

Example 2. To prove: If the sides in each pair of opposite sides of a quadrilateral are congruent, then the angles in a pair of opposite angles of the quadrilateral are congruent.

We draw a sketch that seems to satisfy the hypothesis, label the vertices of the quadrilateral and mark the sketch to indicate the information given in the hypothesis. (The sketch is shown below.)

Plan. (The plan is not part of the proof and you need not write it. Your thinking could follow a different pattern from the one given and still lead to a proof.) Let us choose to prove $\angle B \cong \angle D$. We would expect to prove them congruent by showing that they are corresponding parts of congruent triangles. But there are no triangles in our diagram. We draw $\overline{AC}$ to show which triangles we shall use. Dotting this segment distinguishes it from the parts of the diagram mentioned in the hypothesis.

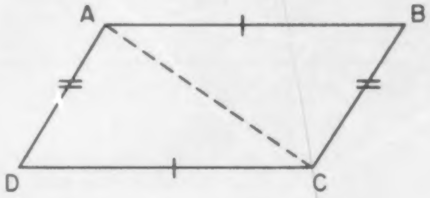
Now in the two triangles we have two pairs of congruent sides, so we would expect to use either the S.A.S. Postulate or the S.S.S. Postulate. To use S.A.S. we would need $\angle B \cong \angle D$. But this is what we are trying to prove so we know we can't use it as part of our proof. That leaves S.S.S. But where is the third pair of congruent sides? In $\triangle BAC$ the third side is $\overline{AC}$. In $\triangle DCA$ the third side is also $\overline{CA}$. Now is our complete plan of proof clear?

We can now write the proof.

Hypothesis: $\overline{AB} \cong \overline{CD}$

$\overline{BC} \cong \overline{DA}$

To prove: $\angle B \cong \angle D$:



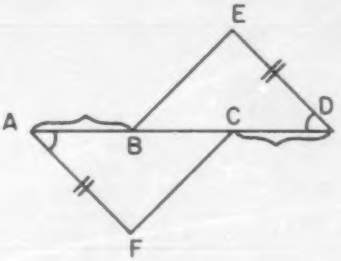
Statement	Reason
1. $\overline{AB} \cong \overline{CD}$	1. Hypothesis
2. $\overline{BC} \cong \overline{DA}$	2. Hypothesis
3. $\overline{CA} \cong \overline{AC}$	3. The reflexive property of congruence
4. $\triangle ABC \cong \triangle CDA$	4. S.S.S. Postulate (1,2,3)
5. $\angle B \cong \angle D$	5. If triangles are congruent (4), then corresponding parts are congruent.

Note in Statement 3 that when we wrote $\overline{CA} \cong \overline{AC}$ we were adhering to the correspondence $ABC \longleftrightarrow CDA$.

__Example 3.__

__Hypothesis:__ A, B, C, D are collinear in that order. E and F are on opposite sides of $\overleftrightarrow{AD}$. $\overline{AB} \cong \overline{DC}$, $\angle A \cong \angle D$, $\overline{AF} \cong \overline{DE}$.

__To prove:__ $\angle F \cong \angle E$.



(__Plan__. We could prove $\angle F \cong \angle E$ by first proving $\triangle FAC \cong \triangle EDB$. In these triangles we are given (1) $\overline{AF} \cong \overline{DE}$ and (2) $\angle A \cong \angle D$. So we can expect to use either the S.A.S. Postulate or the A.S.A. Postulate. Since we are given that $\overline{AB} \cong \overline{DC}$, it seems likely that we shall use the S.A.S. Postulate. Is $\overline{AB}$ a side of $\triangle FAC$? Is $\overline{AC}$? Is $\overline{DC}$ a side of $\triangle EDB$? Is $\overline{DB}$? Then we must try to prove that $\overline{AC} \cong \overline{DB}$. We can do this by using our "betweenness-addition" theorem for segments.)

We continue with the proof.

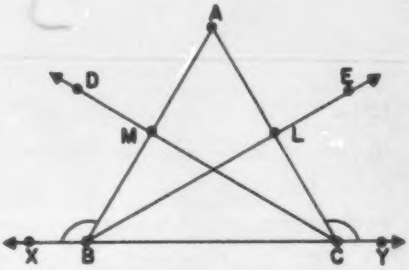
Statement	Reason
1. B and C are between A and D .	1. Hypothesis
2. $\overline{AB} \cong \overline{CD}$	2. Hypothesis
3. $\overline{AC} \cong \overline{BD}$	3. Betweenness-Addition Theorem (1,2)
4. $\angle A \cong \angle D$	4. Hypothesis
5. $\overline{AF} \cong \overline{DE}$	5. Hypothesis
6. $\triangle FAC \cong \triangle EDB$	6. S.A.S. Postulate (3,4,5)
7. $\angle F \cong \angle E$	7. If triangles are congruent (6), then corresponding parts are congruent.

Note the use of names for the theorem in Reason 3 and the postulate in Reason 6, with an indication of the steps which supply the hypothesis of the conditional in each case.

Example 4.

Hypothesis: $\triangle ABC$,
$\overrightarrow{BE}$ is the midray of $\angle ABC$ intersecting $\overline{AC}$ in L .
$\overrightarrow{CD}$ is the midray of $\angle ACB$ intersecting $\overline{AB}$ in M .
X, B, C, Y are collinear in that order. $\angle XBM \cong \angle YCL$.

To prove: $\overline{BL} \cong \overline{CM}$.



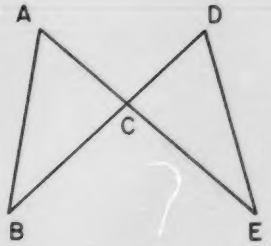
(Plan. We plan to use BCM ⟷ CBL . We can show that $\overline{BC} \cong \overline{CB}$; that $\angle BCL$ (the supplement of $\angle YCA$) $\cong \angle CBM$ (the supplement of $\angle XBA$) ;
that $m\angle BCM$ ($\frac{1}{2}m\angle BCL$) $= m\angle CBL$ ($\frac{1}{2}m\angle CBM$) . This yields A.S.A.)

Statement	Reason
1. $\angle XBM \cong \angle YCL$.	1. Hypothesis.
2. X, B, C are collinear. B, C, Y are collinear.	2. Hypothesis.
3. $\angle MBC \cong \angle LCB$.	3. The supplements of congruent angles are congruent. (1,2)
4. $\overrightarrow{BL}$ is the midray of $\angle MBC$. $\overrightarrow{CM}$ is the midray of $\angle LCB$.	4. Hypothesis.
5. $m \angle MBC = m \angle LCB$.	5. If angles are congruent (3), then they have the same measure.
6. $m \angle MCB = m \angle LBC$.	6. Multiplication property of equality (4,5).
7. $\angle MCB \cong \angle LBC$.	7. If two angles have the same measure (6), then they are congruent.
8. $\overline{BC} \cong \overline{CB}$.	8. Reflexive property of congruence.
9. $\triangle BCM \cong \triangle CBL$.	9. A.S.A. Postulate (3,7,8).
10. $\overline{BL} \cong \overline{CM}$.	10. If triangles are congruent (9), then corresponding parts are congruent.

Problem Set 5-7a

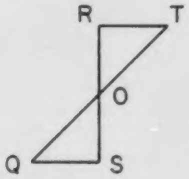
In your proofs of the following problems, mark the given congruences on your diagram.

1. In the figure it is given that $\angle ACB$ and $\angle ECD$ are vertical angles and that $\overline{AC} \cong \overline{DC}$, $\overline{CB} \cong \overline{CE}$. Show (i.e., prove) that $\angle B \cong \angle E$. Copy the following outline and supply the missing reasons including numbers of supporting statements.



Statements	Reasons
1. $\overline{AC} \cong \overline{DC}$.	1. Hypothesis.
2. $\overline{CB} \cong \overline{CE}$.	2. ________________.
3. $\angle ACB$ and $\angle DCE$ are vertical angles.	3. ____________________.
4. $\angle ACB \cong \angle DCE$.	4. If angles are vertical (3), then ____________.
5. $\triangle ACB \cong \triangle DCE$.	5. __________( ),
6. $\angle B \cong \angle E$.	6. If triangles are __________( ), then __________.

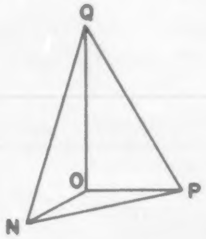
2. In the figure $\angle R$ and $\angle S$ are right angles, and $\overline{QT}$ bisects $\overline{RS}$ at O. Complete the following proof that $\overline{RT} \cong \overline{SQ}$.



Statements	Reasons
1. $\angle R$ and $\angle S$ are right angles.	1. Hypothesis.
2. $\angle R \cong \angle S$.	2. If _____ ( ), then ____.
3. $\overline{QT}$ bisects $\overline{RS}$ at O.	3. Hypothesis.
4. _____ $\cong$ _____.	4. __________( ).

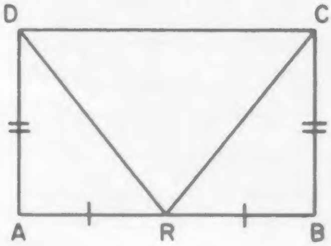
Statements	Reasons
5. $\angle ROT$ and $\angle SOQ$ are ________.	5. If two segments intersect( ), they determine two pairs of vertical angles.
6. $\angle ROT \cong \angle SOQ$.	6. ____angles are congruent( ).
7. $\triangle ROT \cong \triangle SOQ$.	7. ________________( ).
8. ________________.	8. If _____( ), then _______.

3. In the figure $\overline{QO} \perp \overline{NO}$, $\overline{QO} \perp \overline{PO}$, $\overline{NO} \perp \overline{PO}$, and $\overline{NO} \cong \overline{PO}$. Complete the proof that $\overline{NQ} \cong \overline{PQ}$.



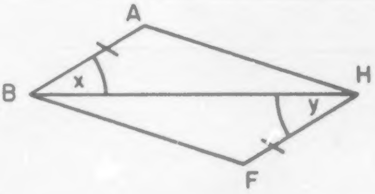
Statements	Reasons
1. $\overline{NO} \cong \overline{PO}$.	1. ______________________.
2. $\overline{QO} \perp \overline{NO}$.	2. Hypothesis.
3. $\angle NOQ$ is a right angle.	3. If lines determined by two segments are perpendicular( ), then ________.
4. $\overline{QO} \perp \overline{PO}$.	4. ______________________.
5. $\angle POQ$ is a right angle.	5. If ______( ), then ______.
6. $\angle NOQ \cong \angle POQ$.	6. Right angles are ____( ).
7. $\overline{QO} \cong \overline{QO}$.	7. ______________________.
8. $\triangle NOQ \cong \triangle POQ$.	8. ______________________( ).
9. $\overline{NQ} \cong \overline{PQ}$.	9. If ______( ), then ______.

4. (a) Quadrilateral ABCD has four right angles and AD = BC . If R is the midpoint of $\overline{AB}$, prove that $\overline{RC} \cong \overline{RD}$.

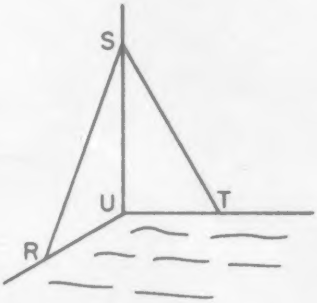


   (b) What pairs of acute angles in the diagram appear congruent? Prove that they are congruent.

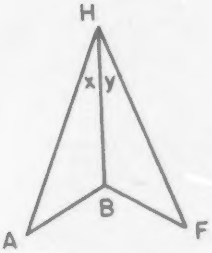
5. In the figure at the right there is a quadrilateral ABFH with AB = FH and $m \angle x = m \angle y$. Prove that $\angle A \cong \angle F$.



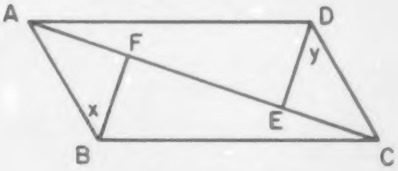
6. Prove that if RS = TS and UR = UT , then $\angle STU \cong \angle SRU$.



7. It is given that $m \angle ABH = m \angle FBH$ and $\angle x \cong \angle y$. Prove that AH = FH .

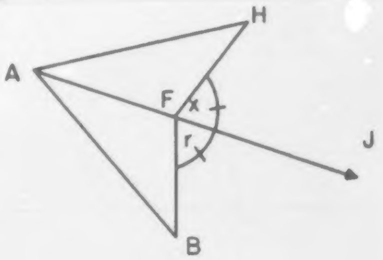


8. Prove that if $\angle BFA$ is a right angle, $\angle DEC$ is a right angle, $m\angle x = m\angle y$ and $BF = DE$, then $FA = EC$.

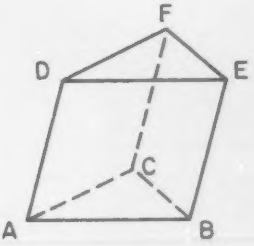


9. Prove that if segment $\overline{AR}$ and segment $\overline{BH}$ bisect each other at point F, then $\overline{AH} \cong \overline{BR}$. (In Example 1 we proved $\overline{AB} \cong \overline{RH}$.)

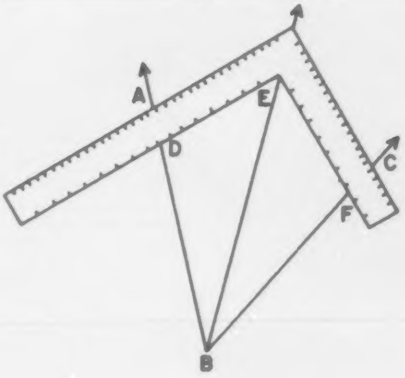
10. Suppose that in this figure $\overrightarrow{AJ}$ bisects $\angle HAB$ and $\angle x \cong \angle r$. Prove that $FH = FB$.



11. If $\overline{DF} \cong \overline{AC}$, $\overline{FE} \cong \overline{CB}$ and $\overline{DE} \cong \overline{AB}$, show that $m\angle DFE = m\angle ACB$.

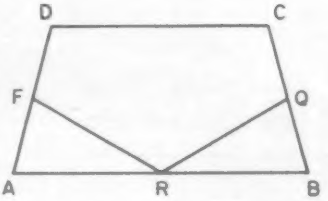


12. A carpenter may bisect an angle using his steel square as follows: Mark off D on $\overrightarrow{BA}$ and F on $\overrightarrow{BC}$ such that $BD = BF$. Then adjust the square so that $ED = EF$ as shown. Prove that $\overrightarrow{BE}$ bisects $\angle ABC$.



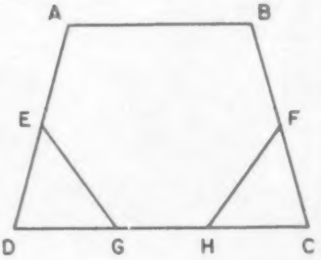
13. Given quadrilateral ABCD with AD = BC and $\angle A \cong \angle B$.
R is the midpoint of $\overline{AB}$,
F is a point between A and D ,
Q is a point between B and C ,
$\overline{DF} \cong \overline{CQ}$.

Prove that $\overline{RF} \cong \overline{RQ}$.



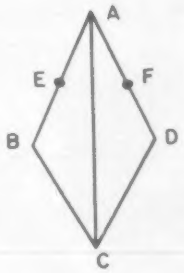
14. Given quadrilateral ABCD .
E is the midpoint of $\overline{AD}$,
F is the midpoint of $\overline{BC}$,
D, G, H, C are collinear in that order, $\overline{CG} \cong \overline{DH}$,
$\overline{AD} \cong \overline{BC}$, $\angle EDG \cong \angle FCH$.

Prove that $\overline{EG} \cong \overline{FH}$.



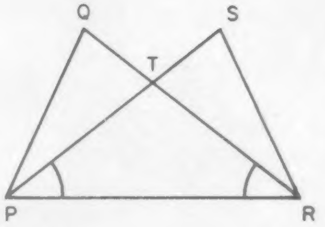
15. Given quadrilateral ABCD .
E is the midpoint of $\overline{AB}$,
F is the midpoint of $\overline{AD}$,
$\angle EAC \cong \angle FAC$, $\overline{AE} \cong \overline{AF}$.

Prove that $\triangle BAC \cong \triangle DAC$.



*16. Hypothesis: $\angle SPR \cong \angle QRP$,
$\overline{SP} \cong \overline{QR}$,
$\overline{QR}$ and $\overline{SP}$
intersect in T.

To prove: $\overline{SR} \cong \overline{QP}$.



Plan: The figure shows two triangles that contain $\overline{SR}$ as a side, and two triangles that contain $\overline{QP}$ as a side. Name them. The two correspondences that we might consider are:

$$SRT \leftrightarrow QPT \quad \text{and} \quad SRP \leftrightarrow QPR .$$

Let us first consider $SRT \leftrightarrow QPT$. What congruences of corresponding parts are given in ΔSRT and ΔQPT? What congruences can you prove? Are they sufficient to prove the triangles congruent? Now try ΔSRP and ΔQPR. Will the S.A.S. Postulate be useful in proving them congruent? Using this plan write the proof.

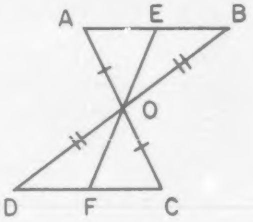
Finding a Proof.

Some students find proofs more easily than others. The reason for this could be that they follow some systematic procedure in discovering proofs. We present here one procedure which has been found useful.

Example.

Hypothesis: $\overline{AC}$, $\overline{EF}$, and $\overline{BD}$ are concurrent at O. $\overline{OA} \cong \overline{OC}$; $\overline{OB} \cong \overline{OD}$. A, E, B are collinear in that order. D, F, C are collinear in that order.

To prove: $\overline{OE} \cong \overline{OF}$.



This method consists of writing (or thinking) two columns headed as follows:

	I can prove		If I can prove		
1.	$\overline{EO} \cong \overline{FO}$.	1.	$\Delta EOA \cong \Delta FOC$		?
2.	$\Delta EOA \cong \Delta FOC$.	2.	$\angle EOA \cong \angle FOC$,	(A)	✓
			$\overline{OA} \cong \overline{OC}$,	(S)	✓
			$\angle OAE \cong \angle OCF$.	(A)	?
3.	$\angle OAE \cong \angle OCF$.	3.	$\Delta OAB \cong \Delta OCD$.		?
4.	$\Delta OAB \cong \Delta OCD$.	4.	$\overline{OA} \cong \overline{OC}$.		✓
			$\angle AOB \cong \angle COD$,		✓
			$\overline{OB} \cong \overline{OD}$.		✓

We read Line 1 as follows: I can prove $\overline{EO} \cong \overline{FO}$ if I can prove $\Delta EOA \cong \Delta FOC$.

We read Line 2 as follows: I can prove $\Delta EOA \cong \Delta FOC$ if I can prove (a) $\angle EOA \cong \angle FOC$, (b) $\overline{OA} \cong \overline{OC}$, (c) $\angle OAE \cong \angle OCF$. Because (a) and (b) are given or can easily be proved they are checked. Statement (c) then is to be considered. It is brought down to Line 3. Every statement in the second column that has a "?" is brought down to the next line in Column 1. In this manner we finally arrive at the end where every statement is checked.

We can now write the proof by reversing the order of the statements in the second column, as follows:

	Statements		Reasons
1.	$\overline{OB} \cong \overline{OD}$.	1.	Hypothesis.
2.	$\angle AOB \cong \angle COD$.	2.	Vertical angles are congruent.
3.	$\overline{OA} \cong \overline{OC}$.	3.	Hypothesis.
4.	$\Delta OAB \cong \Delta OCD$.	4.	S.A.S. Postulate (1,2,3).
5.	$\angle OAE \cong \angle OCF$.	5.	Definition of congruence. (4)
6.	$\overline{OA} \cong \overline{OC}$.	6.	Hypothesis.
7.	$\angle EOA \cong \angle FOC$.	7.	Vertical angles are congruent.
8.	$\Delta EOA \cong \Delta FOC$.	8.	A.S.A. Postulate (5,6,7).
9.	$\overline{EO} \cong \overline{FO}$.	9.	Definition of congruence. (8)

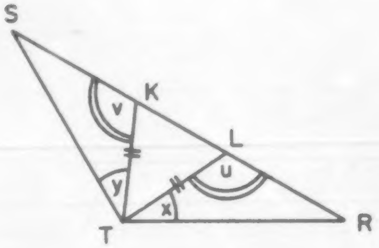
You will note that some of the reasons are written in abbreviated form. This is done when we are quite certain that we know the complete statement and find it convenient to use an abbreviation.

Problem Set 5-7b

1. As represented in the figure at the right, $\angle x \cong \angle y$, $\angle u \cong \angle v$, $TK = TL$, and S , K , L , R are collinear.

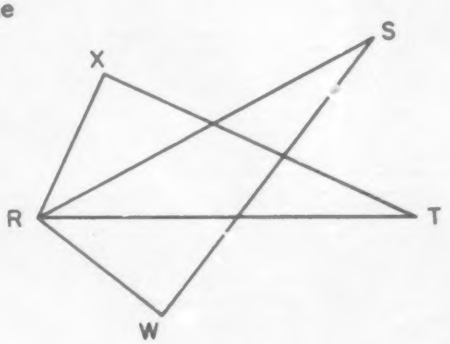
   Prove that

   (a) $SK = RL$, and
   (b) $\Delta RTK \cong \Delta STL$.



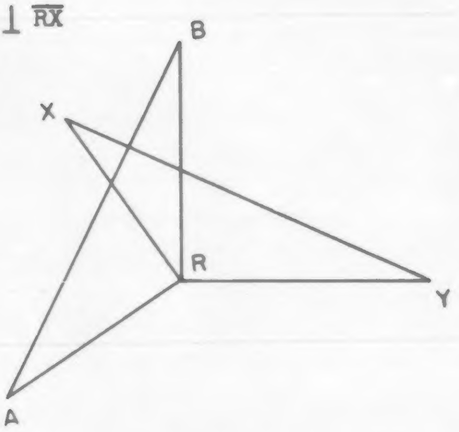
2. In the plane figure at the right we have given that $\overline{RS} \cong \overline{RT}$, $\overline{RW} \cong \overline{RX}$, and $\angle SRX \cong \angle TRW$.

   Prove that $\angle X \cong \angle W$.



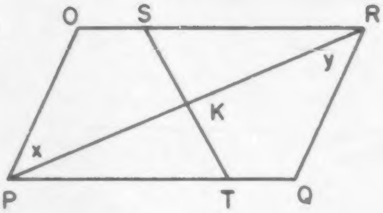
3. In the plane figure $\overline{AR} \perp \overline{RX}$, $\overline{BR} \perp \overline{RY}$, $AR = RX$, and $BR = RY$.

   Prove that $AB = XY$.

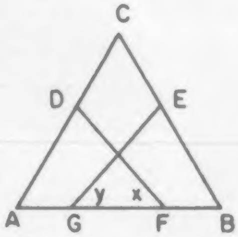


4. In quadrilateral ORQP , we have given that $\angle OPQ \cong \angle ORQ$, $\angle x \cong \angle y$, and $\overline{ST}$ bisects $\overline{PR}$ at K .

   Prove that RS = PT .

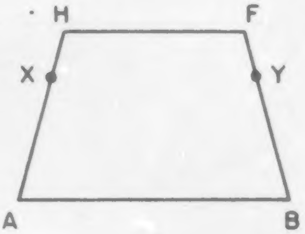


5. Points D, E, F, G are on the sides of $\triangle ABC$ as in the diagram. Prove that if $\angle A \cong \angle B$, $\angle x \cong \angle y$, and $\overline{AG} \cong \overline{FB}$, then $\overline{DF} \cong \overline{EG}$.

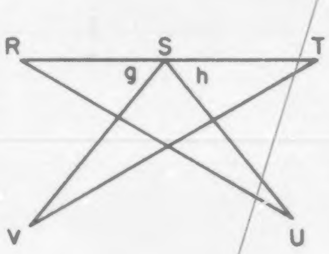


6. ABFH is a quadrilateral with AH = BF and $\angle A \cong \angle B$. X, Y are on $\overline{AH}$ and $\overline{BF}$, respectively.

   Prove that if XH = YF , then $\overline{AY} \cong \overline{BX}$.

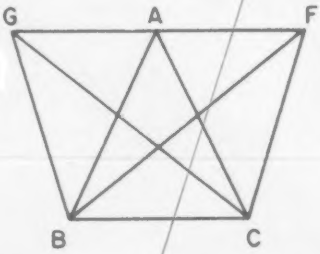


7. In this plane figure prove that if S is the midpoint of $\overline{RT}$, SU = SV , and $\angle g \cong \angle h$, then $\triangle RSU \cong \triangle TSV$ and $\angle V \cong \angle U$.

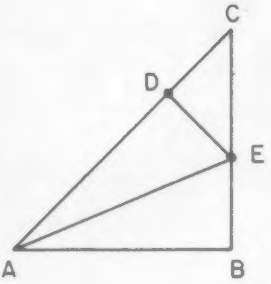


8. In the figure, we have given that BG = CF , A is the midpoint of $\overline{GF}$, and AB = AC .

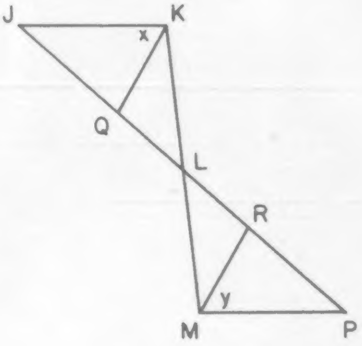
   Prove that $\overline{BF} \cong \overline{CG}$.



9. In the accompanying figure, $AD = AB$, $EB = ED$, and $\angle B$ is a right angle. Prove that $\angle EDA$ is a right angle, and that $\overrightarrow{AE}$ bisects $\angle DAB$.



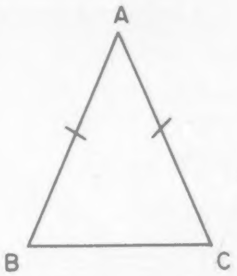
10. In the accompanying figure, the segments $\overline{JP}$ and $\overline{KM}$ bisect each other at L . We are given that $\angle x \cong \angle y$. Prove that $\overline{KQ} \cong \overline{MR}$.



11. In quadrilateral $ABCD$, $AD = BC$, $\angle A \cong \angle B$, R is the midpoint of $\overline{AB}$. Let E, F be points on $\overline{CD}$ such that $EC = FD$. Prove that $RE = RF$.

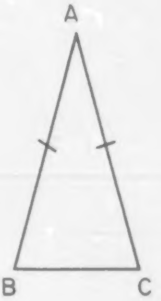
5-8. Isosceles Triangle Theorem.

Let us now consider the interesting case of $\triangle ABC$ in which $\overline{AB} \cong \overline{AC}$ and the correspondence of the triangle to itself given in $ABC \longleftrightarrow ACB$. The corresponding sides then are $\overline{AB}$ and $\overline{AC}$, $\overline{AC}$ and $\overline{AB}$, $\overline{BC}$ and $\overline{CB}$.



Since the sides in each pair are congruent we can say that $\triangle ABC \cong \triangle ACB$. Why? It follows that corresponding angles, for instance, $\angle B$ and $\angle C$, are congruent. This suggests

THEOREM 5-6. If two sides of a triangle are congruent, then the angles opposite these sides are congruent.



Proof: As in the accompanying figure let $\overline{AB}$ and $\overline{AC}$ denote the congruent sides. In terms of this notation we have the

Hypothesis: $\overline{AB} \cong \overline{AC}$. We are are required

To prove: $\angle B \cong \angle C$.

Statements	Reasons
1. $\overline{AB} \cong \overline{AC}$.	1. Hypothesis.
2. $\overline{AC} \cong \overline{AB}$.	2. Symmetric property of congruence.
3. $\overline{BC} \cong \overline{CB}$.	3. Reflexive property of congruence.
4. $\triangle ABC \cong \triangle ACB$.	4. S.S.S. Postulate. (1,2,3)
5. $\angle B \cong \angle C$.	5. Definition of congruence. (4)

Can you also prove this theorem using the S.A.S. Postulate?

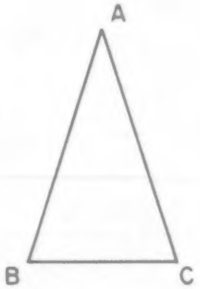
In this chapter we learned how to write proofs in the two-column form. In previous chapters we wrote proofs in paragraphs or essay form. You should be able to write a proof in either form.

In the two-column form it is easy to check that each statement in the proof is supported by a reason. In the paragraph form it is often easier to communicate the plan of proof. It is wise to check a paragraph proof by listing in one column each statement made and in another column the supporting reason for the statement. This in fact should be the corresponding two-column proof.

It will be helpful if the same proof is seen in both forms. We therefore present a proof in paragraph form for Theorem 5-6 for which we just presented a two-column proof; we omit some phrases which you are to supply.

Theorem 5-6. If two sides of a triangle are congruent, then the angles opposite these sides are congruent.

Proof: Let ABC denote the triangle and let $\overline{AB}$ and $\overline{AC}$ be the two congruent sides. Then in the correspondence $ABC \longleftrightarrow ACB$ we know that $\overline{BC} \cong \overline{CB}$ because ____________. Since $\overline{AB} \cong \overline{AC}$ and $\overline{AC} \cong \overline{AB}$ we conclude that $\triangle ABC \cong$ __________ by the ____________ Postulate. It therefore follows that $\angle B \cong \angle C$ because __________, and this completes the proof.



DEFINITIONS. A triangle with (at least) two congruent sides is called isosceles.

If two sides are congruent, then the remaining side is called the base and the angle included between the congruent sides is called the vertex angle.

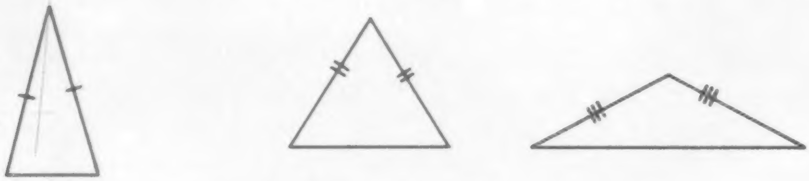
The angles that include the base are called the base angles.

If a triangle has three congruent sides, then any side may be considered as a base of the triangle. The angle opposite a base is considered the vertex corresponding to that base, and the angles that include the base are called the base angles corresponding to that base. In these terms we can restate Theorem 5-6 in this form: "The base angles of an isosceles triangle are congruent."

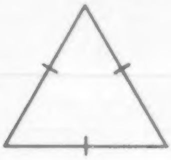
DEFINITIONS. A triangle with three congruent sides is called equilateral.

A triangle with three congruent angles is called equiangular.

Below are several pictures of isosceles triangles and one of an equilateral triangle. Is every equilateral triangle isosceles? Is every isosceles triangle equilateral?



Isosceles Triangles



Equilateral Triangle

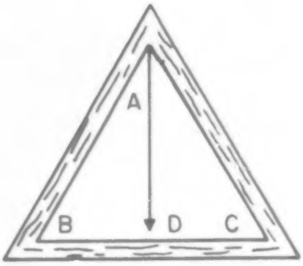
From Theorem 5-6 we can easily deduce that every equilateral triangle is equiangular. In a case such as this, a theorem which follows easily from another is often called a corollary of it.

Corollary 5-6-1. Every equilateral triangle is equiangular.

Problem Set 5-8

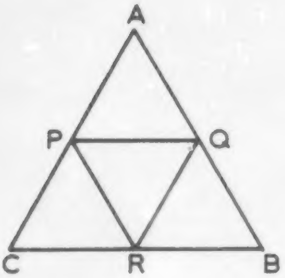
*1. Prove Corollary 5-6-1.

2. Before the modern spirit level was invented, the plumb level was used. It was made by constructing a wooden or metal isosceles triangle ABC and hanging a weight D from the vertex between the congruent sides. Prove that when D is at the midpoint of $\overline{BC}$, the base of the triangle is level.



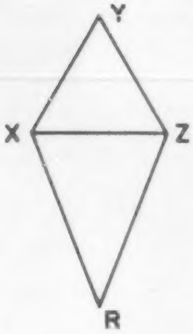
(In this case, level means at right angles to $\overline{AD}$.)

3.



Given equilateral triangle ABC with Q, R, and P, the midpoints of the sides as shown. Prove that triangle PQR is equilateral.

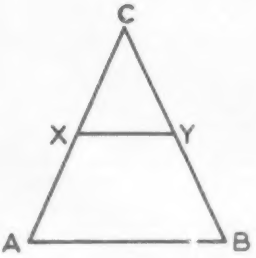
4.



In quadrilateral XYZR, $\overline{XY} \cong \overline{ZY}$
$\overline{XR} \cong \overline{ZR}$

Prove $\angle YXR \cong \angle YZR$.

5.



Write a proof in two-column form, then a proof in paragraph form, for the following statement:

If X and Y are the midpoints of the congruent sides $\overline{AC}$ and $\overline{BC}$ of isosceles triangle ABC, then $\angle CXY \cong \angle CYX$.

6. Let P be any point on the bisector of the vertex angle of an isosceles triangle. Prove that P is equidistant from the endpoints of the base.

7. Prove that the segments which bisect the base angles of an isosceles triangle and end in the opposite sides are congruent.

8. Prove that, if in two isosceles triangles the base and a base angle of one have the same measure as the base and a base angle of the other respectively, then the triangles are congruent.

5-9. *Converses*.

In order that you may understand better the relationship of our next theorem to our last theorem we pause briefly to illustrate the idea of converse statements. Let us write separately the hypothesis and the conclusion of the statement, "If I live in Chicago, then I live in Illinois."

Hypothesis: I live in Chicago.
Conclusion: I live in Illinois.

We can form a related statement by interchanging the hypothesis and the conclusion. The new statement is:

If I live in Illinois, then I live in Chicago.

Such a related statement is called the converse of the original statement.

As you can see from this example, the converse of a statement need not be true even though the statement is true. The following four geometrical statements illustrate the various possibilities as regards the validity of a statement and its converse. We consider a statement to be *valid* if it can be deduced from the postulates and definitions in our formal geometry.

Example 1. Consider the statement:

Vertical angles are congruent.

Let us write it in the "if-then" form.

Statement: If $\angle A$ and $\angle B$ are vertical angles, then $\angle A \cong \angle B$.

It is now simple to write its converse.

Converse: If $\angle A \cong \angle B$, then $\angle A$ and $\angle B$ are vertical angles.

Is the statement valid? Is the converse valid? You can answer the first of these questions by noting Theorem 4-19. Let us now consider the converse. If we can show one instance of two congruent angles that are not vertical, then the converse is not valid. Is the converse valid?

Example 2. Consider the statement:

Statement: If two angles are congruent, then they are right angles.

Is this statement valid? Now let us consider its converse.

Converse: If two angles are right angles, then they are congruent.

Is this converse valid?

Example 3.

Statement: If $\overline{AB} \cong \overline{CD}$, then $AB = CD$.

Is this valid?

Converse: If $AB = CD$, then $\overline{AB} \cong \overline{CD}$.

Is this valid?

Example 4.

Statement: If $\triangle ABC \cong \triangle DEF$, then $\angle A \cong \angle B$.

Is this valid?

Converse: If $\angle A \cong \angle B$, then $\triangle ABC \cong \triangle DEF$.

Is this converse valid?

Considering these four examples we see that a statement and its converse may both be valid, or both be not valid, or one valid and the other not valid. We therefore observe that in general, a statement and its converse need not agree as regards their validity.

Notice that the parts of a definition in complete form are converses of each other, and both are accepted as valid.

Now consider the statement:

> The base angles of an isosceles triangle are congruent.

Let us write it in the "If-then" form:

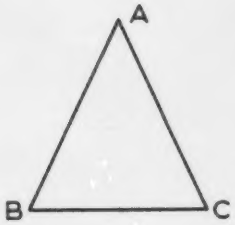
> If two sides of a triangle are congruent, then the angles opposite these sides are congruent.

This statement, of course, is Theorem 5-6. State its converse. Do you think that this converse is necessarily valid simply because it is the converse of a valid statement?

The converse of Theorem 5-6 is indeed valid and we present it as Theorem 5-7.

THEOREM 5-7. If two angles of a triangle are congruent, then the sides opposite these angles are congruent.

Proof: Let A, B, C denote the vertices of the triangle. Then we have



Hypothesis: $\angle B \cong \angle C$.

To prove: $\overline{AB} \cong \overline{AC}$.

Statements	Reasons
1. $\angle B \cong \angle C$.	1. Hypothesis.
2. $\overline{BC} \cong \overline{CB}$.	2. Reflexive property of congruence.
3. $\angle C \cong \angle B$.	3. Symmetric property of congruence.
4. $\Delta ABC \cong \Delta ACB$.	4. A.S.A. Postulate (1,2,3).
5. $\overline{AC} \cong \overline{AB}$.	5. Definition of congruence. (4)

Corollary 5-7-1. Every equiangular triangle is equilateral.

The proof of this corollary is left for you as a problem.

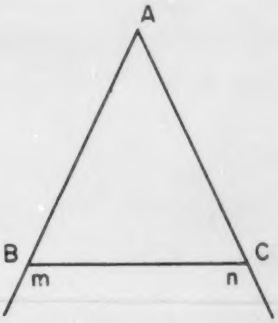
Problem Set 5-9

1. Rewrite each of the following statements in "If _____, then ______" form, and then write its converse. In each case, state whether you think the given statement is valid; whether you think the converse is valid. Give justifications if you can.

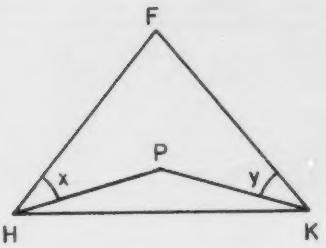
   (a) A positive number has one positive square root.
   (b) The points of the interior of a triangle form a convex set.
   (c) The sum of two odd numbers is odd.
   (d) Right angles are congruent.

*2. Prove Corollary 5-7-1.

3. In the adjacent figure, $\angle m \cong \angle n$. Prove that $\triangle ABC$ is isosceles.

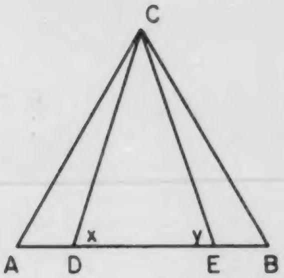


4. In the adjacent figure, $HF = KF$ and $HP = KP$. Prove that $\angle x \cong \angle y$.



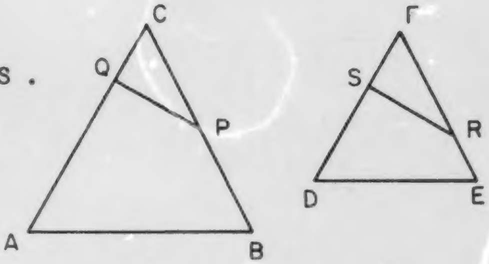
5. In the adjacent figure, $\angle x \cong \angle y$, A, D, E, B are collinear, and $AD = BE$.

Prove that $\triangle ABC$ is isosceles.



6. In the adjacent figure, $\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$, $\overline{PQ} \perp \overline{AC}$, $\overline{RS} \perp \overline{DF}$, and $CQ = FS$.

Prove that $CP = FR$.



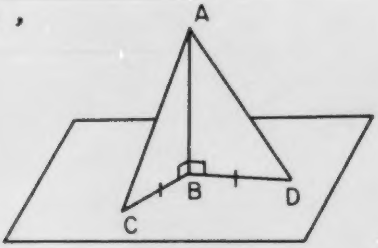
*7. Prove that the ray which bisects the vertex angle of an isosceles triangle is perpendicular to the base of the triangle and also bisects the base.

5-10. Proving Non-Coplanar Triangles Congruent.

Example 1.

Hypothesis: $\triangle CBA$ and $\triangle DBA$,
$\overline{CB} \cong \overline{DB}$,
$\overline{AB} \perp \overline{BD}$,
$\overline{AB} \perp \overline{BC}$.

To prove: $\triangle CBA \cong \triangle DBA$.

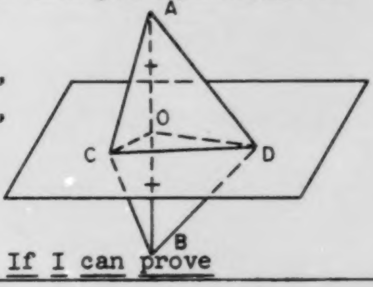


(The triangles we are going to prove congruent are not necessarily coplanar. It is advisable in such cases to see the triangles in perspective. The plan of this proof becomes clear if we are aware that there is a pair of right angles, indicated by the right angle marks, and that $\overline{AB}$ is a side in both triangles. Which congruence postulate should we use? Supply the missing reasons.)

Statements	Reasons
1. $\overline{AB} \cong \overline{AB}$.	1. ____________________.
2. $\angle CBA$ and $\angle DBA$ are right angles.	2. ____________________.
3. $\angle CBA \cong \angle DBA$.	3. ____________________.
4. $\overline{CB} \cong \overline{DB}$.	4. ____________________.
5. $\triangle CBA \cong \triangle DBA$.	5. ____________________.

Example 2.

Below we use the method for finding a proof that was used in Section 5-7. Read it, and when you understand it write the proof.



Hypothesis: $\overline{AB} \perp \overline{CO}$ at O ,
$\overline{AB} \perp \overline{DO}$ at O ,
$\overline{AO} \cong \overline{BO}$.

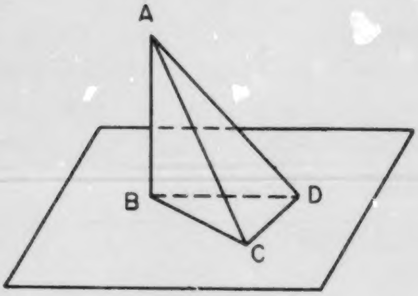
To prove: $\triangle ACD \cong \triangle BCD$.

I can prove	If I can prove
1. $\triangle ACD \cong \triangle BCD$.	1. $\overline{CD} \cong \overline{CD}$, (S) $\overline{AC} \cong \overline{BC}$, (S) ? $\overline{AD} \cong \overline{BD}$. ?
2. $\overline{AC} \cong \overline{BC}$.	2. $\triangle ACO \cong \triangle BCO$. S.A.S.
3. $\overline{AD} \cong \overline{BD}$.	3. $\triangle ADO \cong \triangle BDO$. S.A.S.

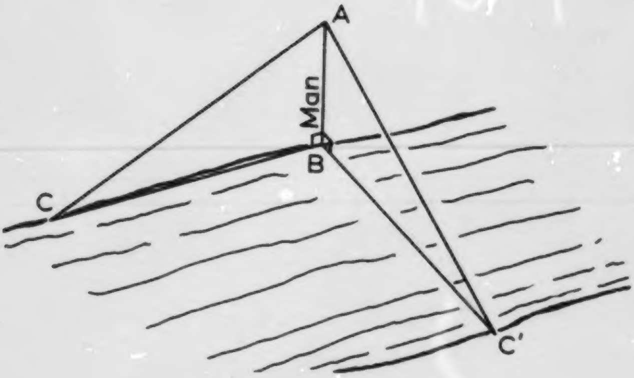
Problem Set 5-10

1. In the figure,
$\overline{AB} \perp \overline{BC}$, $\overline{AB} \perp \overline{BD}$,
and $\overline{BC} \cong \overline{BD}$.

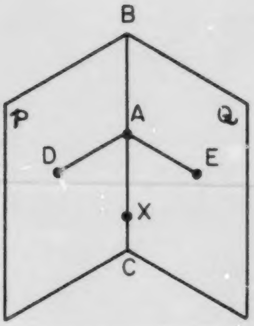
Prove that $\angle ACD \cong \angle ADC$.



2. Napoleon's forces, marching into enemy territory, came upon a stream whose width they did not know. Although the engineers were in the rear, nevertheless, the impetuous commander demanded of his officers to find the width of the river. A young officer immediately stood erect on the bank and pulled the visor of his cap down over his eyes until his line of vision was on the opposite shore. He then turned and sighted along the shore and noted the point where his visor rested. He then paced off this distance along the shore. Was this distance the width of the river? What two triangles were congruent? Why?



3. In dihedral angle D-BC-E ,
$\overline{DA}$ is in plane P ,
$\overline{EA}$ is in plane Q ,
$\overline{DA} \perp \overline{BC}$, $\overline{EA} \perp \overline{BC}$,
$\overline{DA} \cong \overline{EA}$, and X is in $\overline{BC}$.
Show that $\overline{DX} \cong \overline{EX}$.



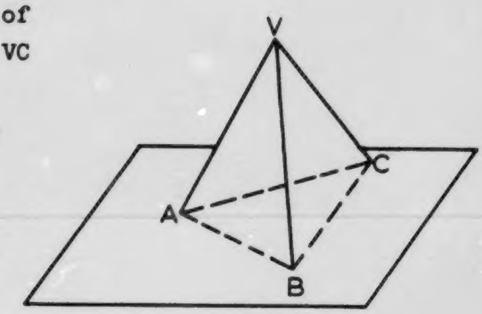
4. Let ABC be any triangle and D a point not in the plane of this triangle. The set consisting of the union of six segments $\overline{AB}$, $\overline{AC}$, $\overline{BC}$, $\overline{AD}$, $\overline{BD}$, $\overline{CD}$ we shall call a skeleton of a tetrahedron. Each of the six segments is called an edge of the tetrahedron; each of the four points A, B, C, D is a vertex; the union of each triangle formed by three vertices and its interior is a face; each angle of a face is a face angle.

(a) Construct an equilateral skeleton of a tetrahedron with toothpicks and quick-drying glue or with soda straws by threading string through them.

(b) How many faces are there? How many face angles?

(c) Two edges of a tetrahedron are opposite edges if they do not intersect. They are adjacent if they do intersect. If each pair of opposite edges are congruent, are any of the faces congruent? If each pair of adjacent edges are congruent, what kind of triangles are the faces?

5. A tripod with three legs of equal lengths VA , VB , VC stands on a plane.

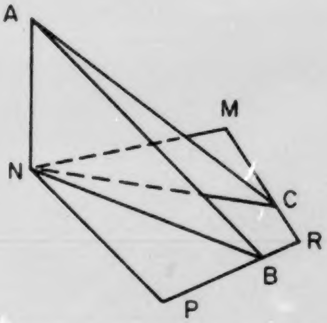


   (a) What can you say, if anything, about the equality of the distances AB , AC , BC ? About the six angles $\angle VAB$, $\angle VAC$, $\angle VBA$, etc.?

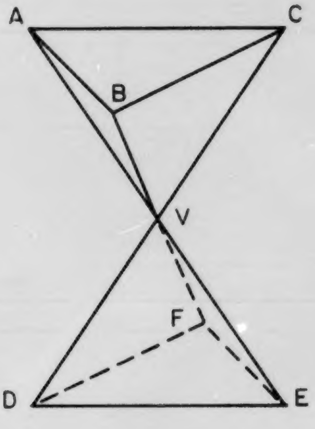
   (b) Answer Part (a) if you are given also that the tripod legs make congruent angles with each other; that is, $\angle AVB \cong \angle BVC \cong \angle AVC$.

6. If in the figure, NP = NM , PR = MR , BR = CR , $\overline{AN} \perp \overline{NB}$, $\overline{AN} \perp \overline{NC}$, $\angle P \cong \angle M$.

   Prove that AB = AC .



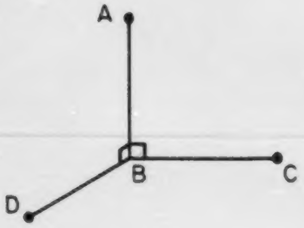
7. Write a proof in paragraph form: If $\overline{AE}$, $\overline{BF}$, and $\overline{CD}$ bisect each other at V then $\Delta ABC \cong \Delta EFD$.



8. Write a proof in paragraph form.

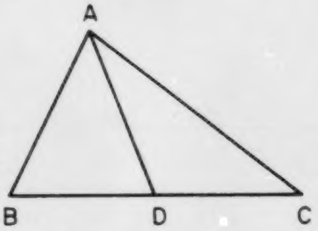
Hypothesis: $\overline{AB} \perp \overline{BC}$, $\overline{CB} \perp \overline{BD}$,
$\overline{DB} \perp \overline{BA}$,
$\overline{AB} \cong \overline{BC} \cong \overline{BD}$.

To prove: $\triangle ADC$ is equiangular.



5-11. *Medians and Angle Bisectors*.

DEFINITION. A *median of a triangle* is a segment whose endpoints are one vertex of the triangle and the midpoint of the opposite side.

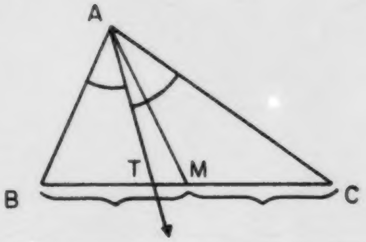


Since a segment has exactly one midpoint there is exactly one median for each vertex of the triangle. In the figure D is the midpoint of $\overline{BC}$. Then $\overline{AD}$ is the median of $\triangle ABC$ from A . How many medians does each triangle have?

THEOREM 5-8. The median to the base of an isosceles triangle (i) bisects the vertex angle and (ii) is perpendicular to the base.

Observe that one portion of the conclusion of this theorem has already been noted in Problem 2 of Problem Set 5-8. The other portion of the conclusion follows in a straightforward manner, so the proof of this theorem is left to you as a problem.

Note that a median from a vertex need not lie on the bisector of the angle at that vertex. In the figure at the right the median from A is the segment $\overline{AM}$ and the bisector of the angle at A is the ray $\overrightarrow{AT}$.



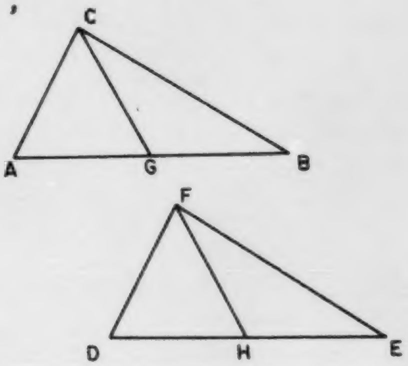
The following theorem has already been proved (Problem 7 of Problem Set 5-9.)

THEOREM 5-9. The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to it.

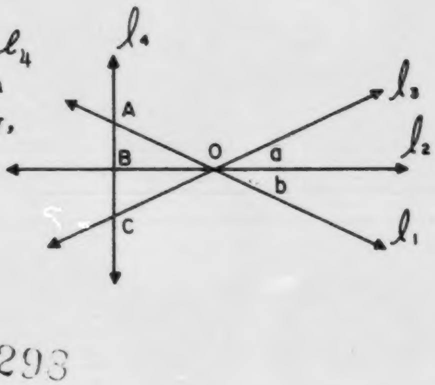
Problem Set 5-11

*1. Prove Theorem 5-8. The median to the base of an isosceles triangle (i) bisects the vertex angle and (ii) is perpendicular to the base.

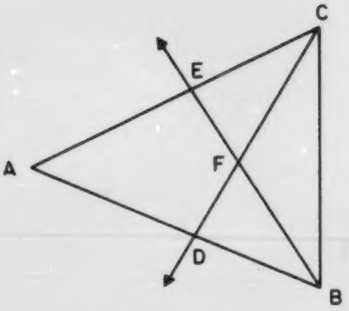
2. Prove that if $\triangle ABC \cong \triangle DEF$, $\overline{CG}$ and $\overline{FH}$ are medians to $\overline{AB}$ and $\overline{DE}$ respectively, then $\triangle ACG \cong \triangle DFH$ and $\overline{CG} \cong \overline{FH}$.



3. In the figure at the right are three concurrent lines, ℓ_1 , ℓ_2 , ℓ_3 , intersecting at O , so that $\angle a \cong \angle b$. Line ℓ_4 intersects ℓ_1 , ℓ_2 , ℓ_3 in points A, B, C, respectively, and $\ell_4 \perp \ell_2$.
Prove: $\overline{OA} \cong \overline{OC}$.



4. In the figure, $\overrightarrow{BE}$ and $\overrightarrow{CD}$ bisect $\angle ABC$ and $\angle BCA$ respectively. $\overline{BA} \cong \overline{CA}$. Prove that $\overline{CE} \cong \overline{BD}$.

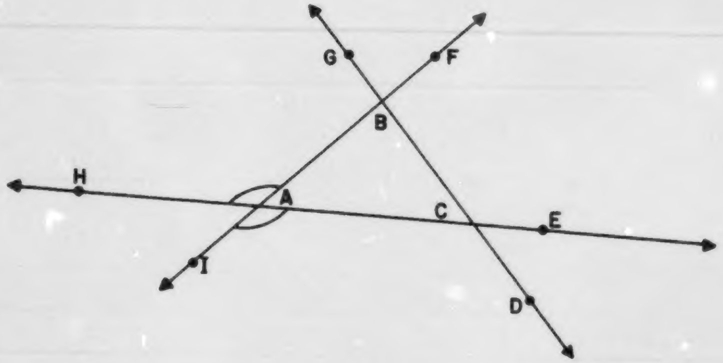


5. Prove: If the median to a side of a triangle is perpendicular to that side, then the triangle is isosceles. [Use a paragraph proof.]

6. Prove: If an angle bisector of a triangle is perpendicular to the opposite side, the triangle must be isosceles. [Use the paragraph form for the proof.]

5-12. Using Congruences as a Mathematical Tool.

Thus far we have used a congruence between triangles to establish a congruence between segments and between angles, and to obtain properties of special triangles, like isosceles and equilateral triangles. We now use a congruence to prove a theorem that will be helpful in our next chapter. But first we consider the definition of an exterior angle of a triangle.



Let A, B, C be vertices of a triangle. Then $\overleftrightarrow{AB}$ and $\angle BAC$ determine a linear pair at A consisting of $\angle BAC$ and $\angle CAI$, as shown above. Similarly $\overleftrightarrow{AC}$ and $\angle BAC$ determine a linear pair at A consisting of $\angle BAC$ and $\angle BAH$. Each angle of the triangle can be considered to be an angle in two linear pairs.

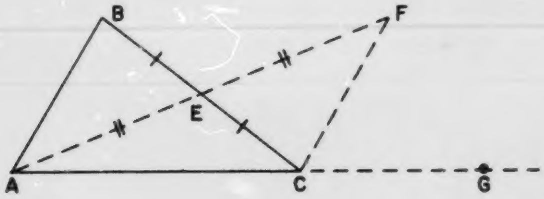
DEFINITION. Each angle of a triangle is sometimes called an interior angle of the triangle.

An angle which forms a linear pair with an interior angle of a triangle is called an exterior angle of the triangle.

Each exterior angle is said to be adjacent to the interior angle with which it forms a linear pair and non-adjacent to the other interior angles of the triangle.

In the figure above $\angle BCE$ is an exterior angle of $\triangle ABC$. It is adjacent to $\angle BCA$ and non-adjacent to $\angle CBA$ and $\angle BAC$. Is $\angle ACD$ also an exterior angle? Does it have the same adjacent angle as $\angle BCE$? The same non-adjacent angles? Name another pair of exterior angles of $\triangle ABC$.

THEOREM 5-10. The measure of an exterior angle of a triangle is greater than the measure of either of its non-adjacent interior angles.



Proof: Let the vertices of the triangle be A, B, and C. Let G be a point on $\overrightarrow{AC}$ such that C is between A and G. Then $\angle BCG$ is an exterior angle of $\triangle ABC$. We must show that $m\angle BCG > m\angle ABC$ and that $m\angle BCG > m\angle BAC$. We first give the argument that $m\angle BCG > m\angle ABC$.

Let E be the midpoint of $\overline{BC}$. Hence $\overline{BE} \cong \overline{CE}$. On ray $\overrightarrow{AE}$ there is a point F such that E is between A and F , and AE = EF . Also $\angle BEA$ and $\angle CEF$ are vertical angles. Consequently $\triangle BEA \cong \triangle CEF$ by S.A.S. Hence, $\angle BCF \cong \angle CBA$. Now Theorem 4-7 tells us that F is an interior point of $\angle BCG$ and, therefore, by Theorem 4-4, $m \angle BCF + m \angle FCG = m \angle BCG$. Since these measures are positive numbers it follows that $m \angle BCG > m \angle BCF$ and therefore $m \angle BCG > m \angle CBA$.

To prove that $m \angle BCG > m \angle BAC$, we use the midpoint of $\overline{AC}$ in the same fashion as in the part above to show that the other exterior angle at C has greater measure than $\angle BAC$. Since the two exterior angles at C are vertical angles, it then follows that $m \angle BCG > m \angle BAC$.

The proof we give for one exterior angle of the triangle can easily be modified to reach the same conclusion for any of the six exterior angles of the triangle.

Theorem 4-21 asserts that in a plane, at each point in a given line there is one and only one line perpendicular to the given line. Because it is a "double-barreled" theorem, we had two statements to prove: that there is a line of the required sort (the existence part) and that there is only one such line (the uniqueness part).

We now use a triangle congruence to prove a companion theorem.

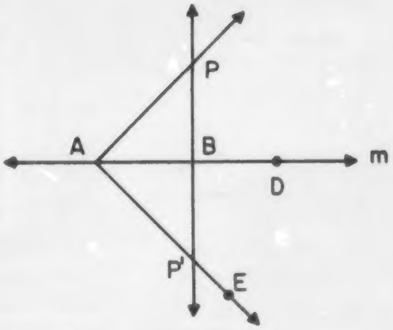
THEOREM 5-11. Given a line and a point not on the line, there is one and only one line which contains the given point and which is perpendicular to the given line.

Proof: Let m be the given line and P the given point not on m . We divide the proof into two parts.

(1) Existence.

We are to prove that there is a line containing P and perpendicular to m .

Let A be a point in m . Then $\overleftrightarrow{PA}$ is either perpendicular to m or not perpendicular to m . If $\overleftrightarrow{PA} \perp m$, Part (i) of our proof is complete. If $\overleftrightarrow{PA}$ is not perpendicular to m , then there is a point D on m such that $m\angle DAP < 90$. Why? Now m separates plane PAD into two halfplanes, one of which contains P . In the other halfplane there is a point E such that $m\angle DAP = m\angle DAE$. On $\overrightarrow{AE}$ there is a point, P' , such that AP' = AP .

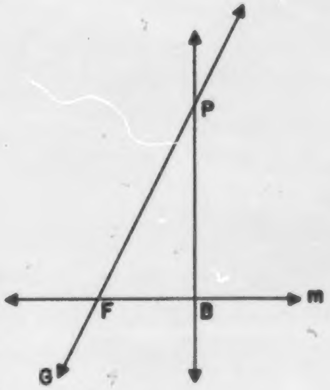


Now we can show that $\overleftrightarrow{PP'}$ is our required line. For $\overleftrightarrow{PP'}$ intersects m . Why? Call the point of intersection B . You should be able to complete the proof of Part (i) using Theorem 5-9.

(ii) Uniqueness.

We are to prove that there cannot be more than one perpendicular to m containing P .

Suppose that $\overleftrightarrow{PG}$ were another perpendicular to m . Then $\overleftrightarrow{PG}$ would intersect m in a point, say F . Why? But this would then give us an exterior angle of $\triangle PFB$ at F whose measure would be greater than 90 . Why? Is this possible? Why? Therefore we cannot have a second perpendicular from P to m and we have proved the uniqueness part of our theorem.

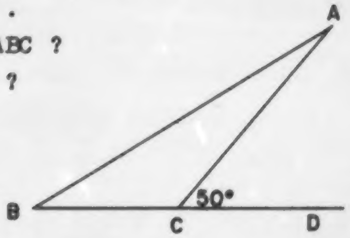


DEFINITION. If P is not in m , and the perpendicular to m containing P intersects m in F , then F is called the foot of the perpendicular from P to m .

Problem Set 5-12a

1. Prove Theorem 5-10 for an exterior angle at B .

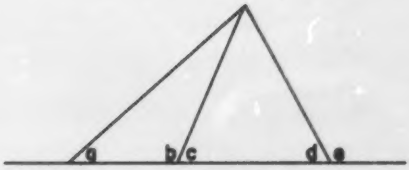
2. In this figure, $m \angle ACD = 50$. What must be true about $m \angle ABC$? About $m \angle BAC$? About $m \angle ACB$?



3. Prove: The exterior angles of an equilateral triangle are congruent.

4. Arrange in order starting with the smallest

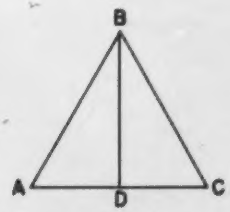
   (a) $m \angle a$, $m \angle c$.
   (b) $m \angle b$, $m \angle d$.
   (c) $m \angle e$, $m \angle c$.
   (d) $m \angle c$, $m \angle a$, $m \angle e$.



5. Hypothesis: $\triangle ABC$ is isosceles.
   $\overline{BD} \perp \overline{AC}$, $\overline{AB} \cong \overline{BC}$.

   Problem: Must $\overline{BD}$ therefore be the median to $\overline{AC}$?

   First, consider the following questions:



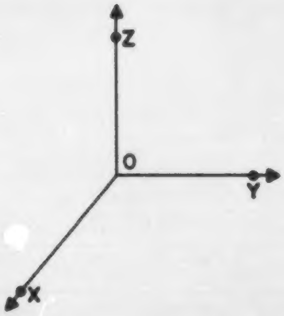
   (a) What do we know about the median drawn to the base of an isosceles triangle? (from Theorem 5-8).
   (b) Why do we know that $\overline{BD}$ and the median cannot be two distinct segments? State the theorem that you used in answering this question.
   (c) Must $\overline{BD}$ therefore be the median?
   (d) State the hypothesis of this problem and the conclusion you have derived in the form of an if _____, then _____ statement.

6. Suppose that m is a line in a plane $\mathcal{R}$ and that P is a point not in $\mathcal{R}$. Does Theorem 5-11 still apply? Make a diagram and explain.

7. $\overline{OY} \perp \overline{OZ}$, $\overline{OY} \perp \overline{OX}$, $\overline{OX} \perp \overline{OZ}$, P is a point in plane XOY, not on $\overline{OY}$, nor on $\overline{OX}$.

   How many lines containing P are $\perp$ to $\overline{OZ}$? How many $\perp$ $\overline{OY}$? To $\overline{OX}$?

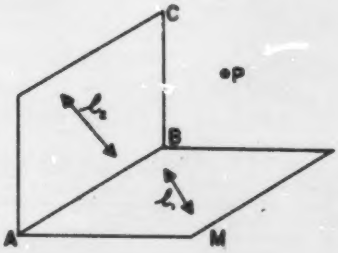
   Make a sketch including all of these perpendiculars. Can you make all the perpendicular lines appear perpendicular?



8. ℓ_1 lies in plane MAB. ℓ_2 lies in plane CAB.

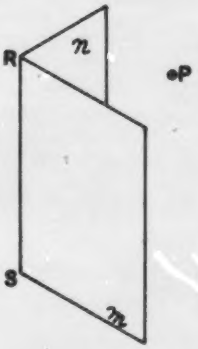
   P is a point in space.

   How many lines containing P are $\perp \ell_1$? How many lines containing P are $\perp \ell_2$?



9. Plane n and plane m intersect in RS. P is a point in space.

   How many lines are in the set of all lines through P $\perp$ to the intersection of plane n and plane m?



10. Is there a unique perpendicular from each point in space to each line in a given plane if the line does not contain the point? If the line does contain the point? Illustrate each with a diagram.

11. Show that there exists no triangle such that two of its angles are right angles.

*12. Show that, if one angle of a triangle is a right angle, then each of the other angles is an acute angle.

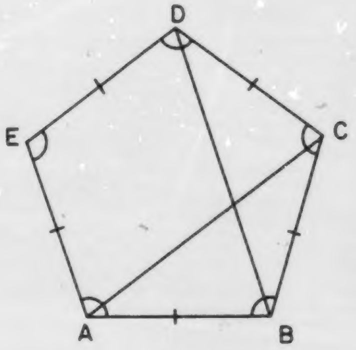
*13. Show that if one angle of a triangle is an obtuse angle, then each of the other angles is an acute angle.

We now give examples of the use of congruences between triangles to derive properties of polygons which are not triangles and of space figures which have polygons as faces.

DEFINITION. A *regular polygon* is a convex polygon all of whose sides are congruent and all of whose angles are congruent.

Example 1.

The diagonals of a regular pentagon are congruent.



Proof: Let ABCDE be a regular pentagon. Then according to the definition of a regular polygon we know that

$\angle A \cong \angle B \cong \angle C \cong \angle D \cong \angle E$,

$\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DE} \cong \overline{EA}$.

We are to prove that

$\overline{AC} \cong \overline{BD} \cong \overline{CE} \cong \overline{DA} \cong \overline{EB}$.

Plan. We prove first that $\overline{AC} \cong \overline{BD}$. Name a triangle of which $\overline{AC}$ is a side; of which $\overline{BD}$ is a side. Set up a correspondence between the vertices that seems to be a congruence. Do you see how to use the S.A.S. Postulate to prove them congruent?

Supply the missing reasons.

Statements	Reasons
1. $\overline{AB} \cong \overline{BC}$.	1. ________________ .
2. $\angle ABC \cong \angle BCD$.	2. ________________ .
3. $\overline{BC} \cong \overline{CD}$.	3. ________________ .
4. $\triangle ABC \cong \triangle BCD$.	4. ________________ .
5. $\overline{AC} \cong \overline{BD}$.	5. ________________ .

Now write a proof that $\overline{AC} \cong \overline{CE}$. Explain how you would go about proving that all five diagonals are congruent.

Example 2.

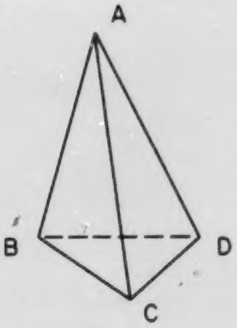
Hypothesis: A, B, C, D are noncoplanar points,
$\overline{AB} \cong \overline{AC} \cong \overline{AD}$,
$\angle BAD \cong \angle DAC \cong \angle CAB$.

To prove: $\triangle BDC$ is equiangular.

Plan. We can prove that $\triangle BDC$ is equiangular if we can prove that $\overline{BD} \cong \overline{DC} \cong \overline{CB}$. We can prove these segments congruent if we can prove that

$\triangle BAD \cong \triangle DAC \cong \triangle CAB$.

These can be proved congruent using the S.A.S. Postulate.

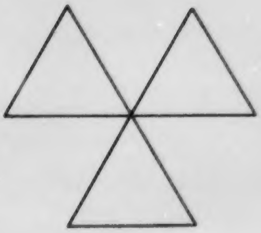


Problem Set 5-12b

1. Prove that the diagonals of a regular quadrilateral are congruent.
2. Do you think that every equilateral quadrilateral is a regular polygon? Justify your answer by an informal argument.

3. Complete the proof for Example 2.

4. The figure at the right is not a picture of a regular polygon. Explain.

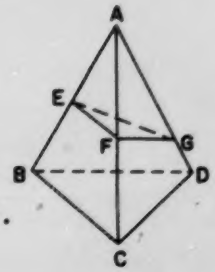


5. ABCDEF is a regular hexagon. Prove that $\triangle BFD$ is an equilateral triangle.

6. ABCDEF is a regular hexagon. Prove that $\overline{AD} \perp \overline{FB}$.

7. The figure at the right is a picture of a space figure in which $\overline{AB} \cong \overline{AC} \cong \overline{AD} \cong \overline{BC} \cong \overline{CD} \cong \overline{BD}$, $\overline{AE} \cong \overline{EB}$, $\overline{AF} \cong \overline{FC}$, and G is in $\overline{AD}$.

Prove that $\triangle EFG$ is an isosceles triangle.



5-13. Summary.

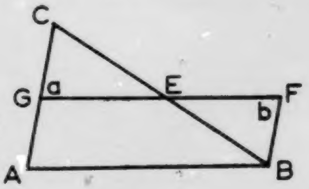
We have investigated congruences between segments, between angles and between triangles in an attempt to gain a deeper insight into the meaning of "same size and shape" as applied to physical objects. A triangle congruence is a one-to-one correspondence between the vertices of one triangle and the vertices of another in which corresponding parts are congruent. The S.A.S., A.S.A., and S.S.S. postulates are remarkable in that they reduce the number of parts we need to consider to establish a congruence between triangles. We learned to plan and write mathematical proofs in two-column and essay forms. We used congruences to prove two isosceles triangle theorems and to help establish some properties of the median and the angle bisector from the vertex angle of an isosceles triangle. Triangle congruences also helped to investigate some properties of regular polygons.

In Chapters VI and VII we shall develop the mathematical machinery for studying the notion of "same shape." The mathematical concept corresponding to the idea of "same shape" is similarity. Just as a congruence between triangles is a correspondence between their vertices (with certain properties), so a similarity between triangles is also a correspondence between their vertices (with other properties). An important concept related to similarity, and an important tool in its development, is the concept of parallel lines which we shall study in Chapter VI.

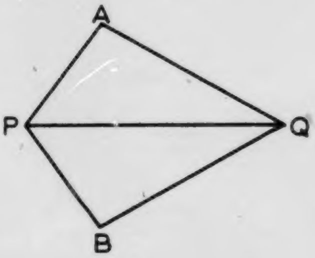
In Chapter 5 we have assumed more than necessary. After we have assumed the S.A.S. Postulate, we can prove the A.S.A. and S.S.S. statements as theorems. We chose not to do so in the text in order to simplify our presentation, but we have included these two proofs in Appendix VI.

Review Problems

1. If $\overline{CB}$ bisects $\overline{GF}$ and $\angle a \cong \angle b$ in the figure, prove that $\overline{GF}$ bisects $\overline{CB}$.

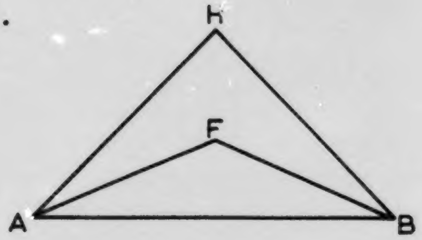


2. If ΔABC is equilateral, prove that $\Delta ABC \cong \Delta CAB$.
3. If the bisector of $\angle G$ in ΔFGH is perpendicular to the opposite side at K, then triangle FGH is isosceles.
4. If $PA = PB$ and $QA = QB$ then $\angle APQ \cong \angle BPQ$. Will the same proof hold regardless of whether A is in the same plane as P, Q, and B?



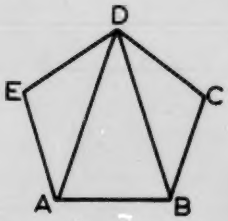
5. In this figure $HA = HB$.
$\overrightarrow{AF}$ bisects $\angle HAB$, and
$\overrightarrow{BF}$ bisects $\angle HBA$.

Prove: $AF = BF$.



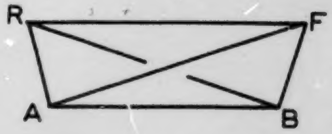
6. ABCDE is a regular pentagon.

Prove that $\angle DAB \cong \angle DBA$.



7. By hypothesis we have given in the figure, that $\overline{AF} \cong \overline{BR}$ and $\overline{AR} \cong \overline{BF}$.

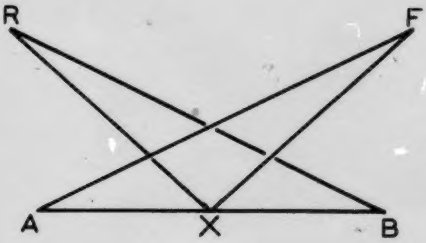
Prove: $\angle ARF \cong \angle BFR$.



(The gap in $\overline{RB}$ was left there so that the figure would not reveal whether or not $\overline{RB}$ intersects AF .) Do you need as part of the hypothesis that the figure likes in a plane?

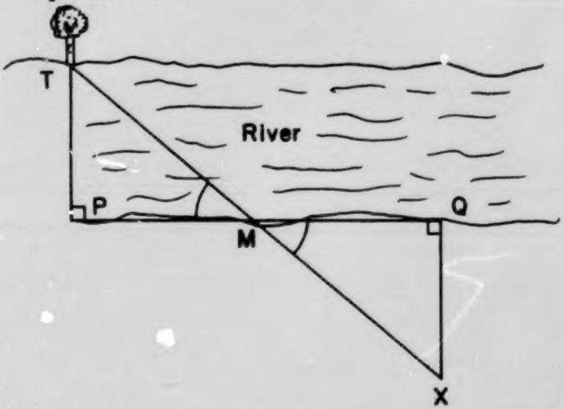
8. (a) If, in the figure, X is the midpoint of $\overline{AB}$, $\angle A \cong \angle B$, and $\angle AXR \cong \angle BXF$, show that $\overline{AF} \cong \overline{BR}$.

(b) Do you need as a part of the hypothesis that the figure lies in a plane?



9. If $\overleftrightarrow{RS}$ is perpendicular to each of three different rays, $\overrightarrow{RA}$, $\overrightarrow{RB}$, $\overrightarrow{RC}$ at R and $RA = RB = RC$, prove that $SA = SB = SC$. (Draw your own figure.)

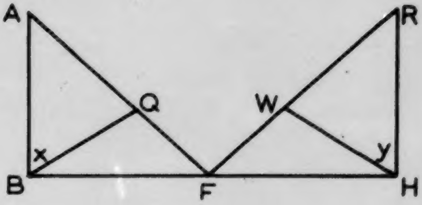
10. A person wishes to find the distance across a river. He does this by sighting a tree, T, on the other side opposite a point P.



Then he paces along a line perpendicular to $\overleftrightarrow{TP}$ to a point M and continues to Q so that $PM = MQ$. Then on a line perpendicular to $\overleftrightarrow{PQ}$ he finds the point X which is collinear with M and T. What other segment in the figure has the same length as $\overline{TP}$? What theorems are used in showing that: $\triangle TPM \cong \triangle XQM$?

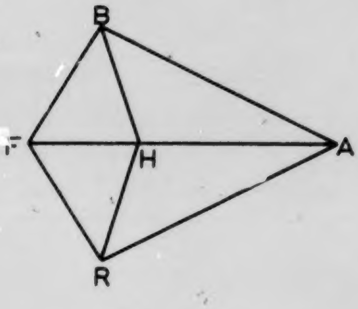
11. In the plane figure, $\overline{AB} \perp \overline{BH}$, $\overline{RH} \perp \overline{BH}$, $\angle x \cong \angle y$, $QB = WH$, and F is the midpoint of $\overline{BH}$

Prove $\triangle BFQ \cong \triangle HFW$.



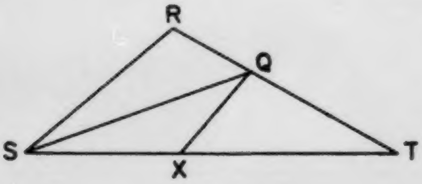
12. In the figure, F, H, A are collinear, $AB = AR$ and $\angle BAH \cong \angle RAH$.

Prove: $FB = FR$.



13. In $\triangle RST$: Point X lies between S and T, and $SX = SR$. Point Q lies between R and T, and SQ bisects $\angle S$. $\overline{QX}$ is drawn.

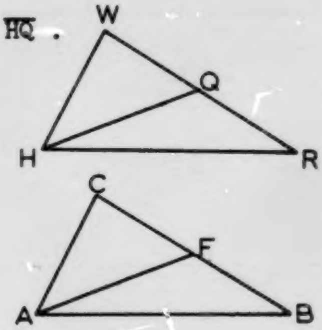
Find an angle congruent to $\angle R$, and prove it.



14. Prove: If two medians of a triangle are perpendicular to their respective sides, then the triangle is equilateral.

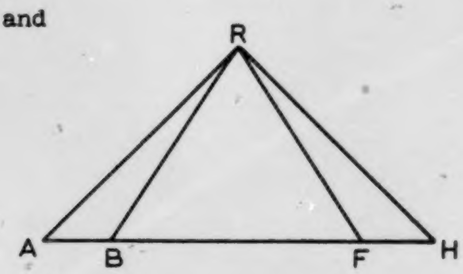
15. In $\triangle ABC$ and $\triangle HRW$, $AB = HR$, $BC = RW$, and median $\overline{AF} \cong$ median $\overline{HQ}$.

Prove that $\triangle ABC \cong \triangle HRW$.



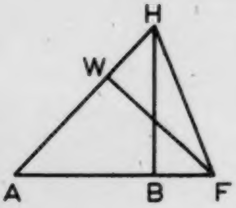
16. In this figure, A, B, F, H are collinear, $AB = HF$ and $RB = RF$.

Prove: $\angle BRH \cong \angle FRA$.



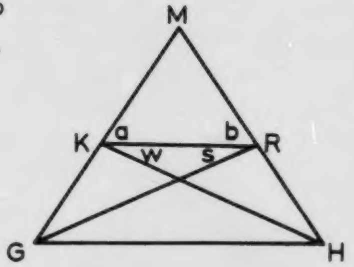
17. In $\triangle HAF$, points B and W are on sides $\overline{AF}$ and $\overline{AH}$, respectively, and $\overline{FW} \perp \overline{AH}$, $\overline{HB} \perp \overline{AF}$, and $AW = AB$.

Prove: $FW = HB$.



18. Hypothesis: In $\triangle MGH$, $\angle a \cong \angle b$ and $\angle w \cong \angle s$ as in the figure.

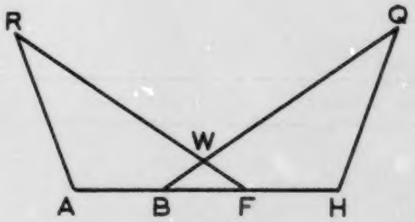
To prove: $\overline{GR} \cong \overline{KH}$.



19. In this figure B and F trisect* $\overline{AH}$, $\angle A \cong \angle H$ and $AR = FQ$.

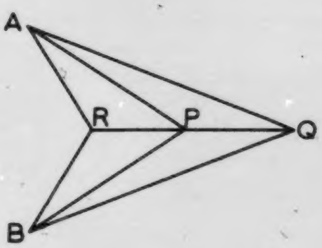
Prove: $BW = FW$.

*Trisect means to separate into three congruent parts.

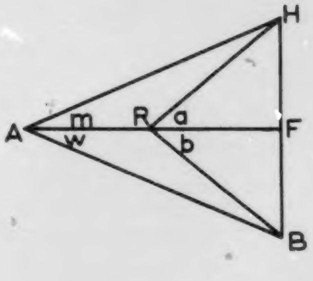


20. (a) Prove: If $PA = PB$, $QA = QB$ and R is on $\overleftrightarrow{PQ}$ as shown in the figure, then $RA = RB$.

(b) Must the five points be coplanar? Will the proof hold whether or not A is in the same plane as B , R , P , and Q ?

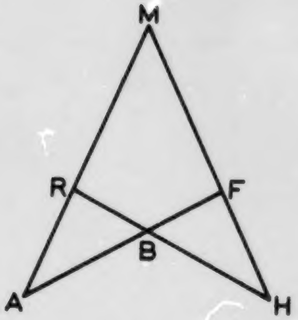


21. If $\angle a \cong \angle b$ and $\angle m \cong \angle w$ in the figure, prove that $\overline{AF} \perp \overline{HB}$.



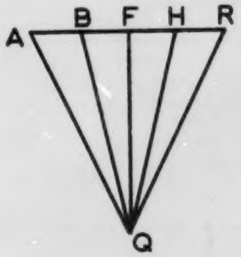
22. $\overline{AF}$ intersects $\overline{RH}$ at B.
$\overleftrightarrow{AR}$ intersects $\overleftrightarrow{HF}$ at M.
$\overline{AB} \cong \overline{HB}$ and $\overline{RB} \cong \overline{FB}$.

Prove: $\angle A \cong \angle H$ and
$\overline{AM} \cong \overline{HM}$.



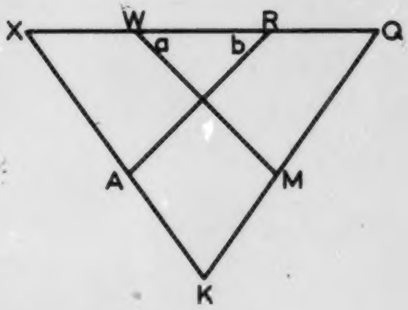
23. In the plane figure, $\overline{FQ} \perp \overline{AR}$,
$\angle AQF \cong \angle RQF$,
$\angle AQB \cong \angle FQB$,
$\angle FQH \cong \angle RQH$.

Prove that $\overline{BQ} \cong \overline{HQ}$.

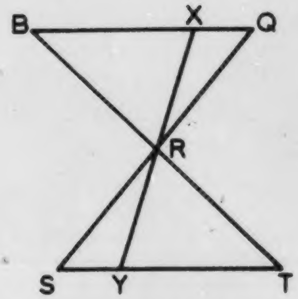


24. In the ΔXKQ , $XW = QR$,
$\angle a \cong \angle b$, $\angle X \cong \angle Q$,

Prove: $KA = KM$.

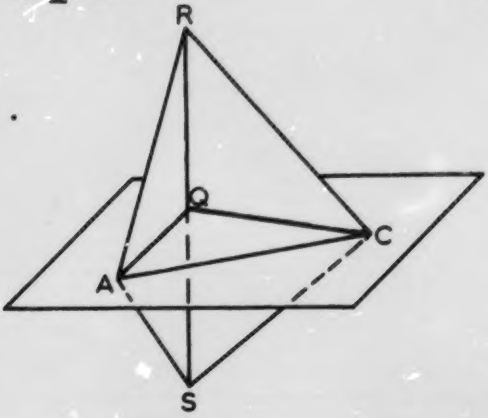


25. Prove that $RX = RY$ if it is given that in the figure:
$BQ = TS$, $m \angle B = m \angle T$ and
$m \angle Q = m \angle S$; X, Y, and R
are collinear.



26. Given: In the figure, $\overline{AQ} \perp \overline{RS}$.
$\overline{RQ} \cong \overline{SQ}$.
$\overline{RC} \cong \overline{SC}$.

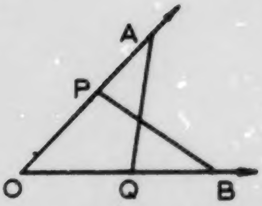
Prove that $\angle RCA \cong \angle SCA$.



27. Hypothesis: $\angle AOB$ with $\overline{OA} \cong \overline{OB}$ and P, Q, points on rays $\overrightarrow{OA}$, $\overrightarrow{OB}$ with $AQ = BP$.

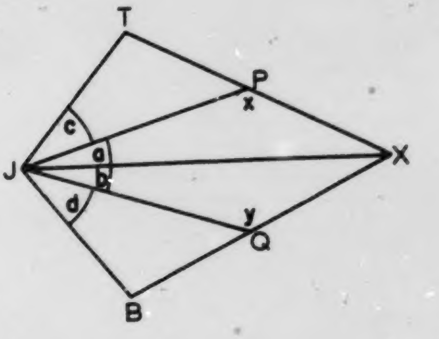
Can the following be proved:

$OP = OQ$?



28. In this plane figure
$\angle a \cong \angle b$, $\angle c \cong \angle d$,
and $JT = JB$.

Prove: $\angle x \cong \angle y$.



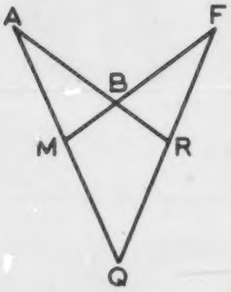
29. Points A, R, S, and C lie on line ℓ in that order. B and D do not lie on ℓ . $AR = CS$, $AB = CD$, $BS = DR$.

(a) Prove that $\angle BSA \cong \angle DRC$.

(b) Need the points A, R, S, C, B, D be coplanar?

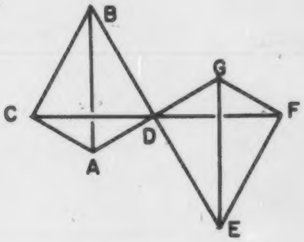
30. In the plane figure, $AB = FB$ and $MB = RB$.

Prove: $\triangle AQR \cong \triangle FQM$.



31. In this figure D is the midpoint of $\overline{AG}$, $\overline{BE}$, and $\overline{CF}$.

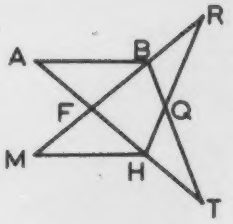
Prove that $\triangle EFG \cong \triangle BCA$.



32. Does the proof for Problem 31 hold even if the segments $\overline{BD}$, $\overline{AD}$, $\overline{CD}$ are not coplanar? Explain.

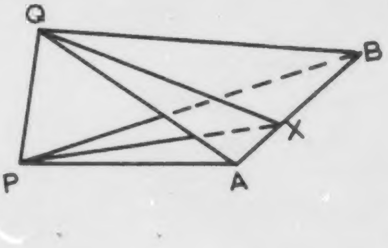
33. In this figure, points F and H trisect $\overline{AT}$, and points F and B trisect $\overline{MR}$. If $AF = FB$, is $\triangle ABT = \triangle MHR$?

Prove your answer.

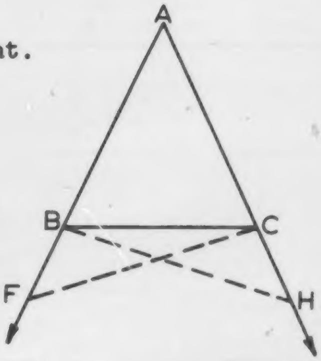


34. Let $\triangle PAB$ and $\triangle QAB$ lie in different planes but have the common side $\overline{AB}$. Let $\triangle PAB \cong \triangle QAB$.

Prove that if X is any point in $\overline{AB}$ then $\triangle PQX$ is isosceles.



35. Complete Euclid's proof of the theorem that the base angles of an isosceles triangle are congruent.



Hypothesis: $AB = AC$.

Prove: $\angle ACB \cong \angle ABC$.

Take a point F with B between A and F , and a point H with C between A and H so that $AH = AF$. Consider ΔACF and ΔABH .

REVIEW PROBLEMS

Chapters 1 to 5

Write the numerals from 1 to 70. Follow each with a "+" or a "-" to indicate whether you consider the statement true or false. True will mean "true under all conditions."

1. A line and a point not on it may be contained in two distinct planes.
2. If a and b are real numbers and $b > a$, then $a - b$ is a negative number.
3. The intersection of $\overrightarrow{AB}$ and $\overrightarrow{BA}$ is $\overline{AB}$.
4. The intersection of every two halfplanes is the interior of an angle.
5. Six correspondences are implied in the statement $\triangle ABC \cong \triangle DEF$.
6. If $\angle RST \cong \angle XYZ$, then $\angle RST = \angle XYZ$.
7. Every plane is a subset of space.
8. If, to prove a statement true, we first assume it to be false, then we are using indirect proof.
9. If a distance AB is three times as great as a distance XY when both are measured in yards, then the distances are equal when both are measured in feet.
10. If $\overline{AB} = \overline{CD}$, then either $A = C$ or $A = D$.
11. A point which belongs to the interior of an angle belongs to the angle.
12. The interior of every triangle is a convex set.
13. Inductive thinking based on experiment is of importance in developing geometry since it often suggests some of the postulates and theorems.

14. A definition in "if-and-only-if" form may be reworded as a statement and its converse.

15. One possible coordinate system is a one-to-one correspondence between the points on a line and the set of real numbers.

16. If a subset of a line is the set of all points whose coordinate x satisfies $-2 \le x \le 5$, then the subset is a segment.

17. A ray coordinate system can be chosen which will assign to a given ray any given number x such that $0 < x < 180$.

18. If the measures of any three parts (sides and angles) of a triangle are given, one and only one triangle which contains parts with those measures is possible.

19. Given $A = \{a,d,e\}$ and $B = \{M,N,e\}$, the intersection of sets A and B is the empty set.

20. The intersection of three planes is the null set.

21. If a, b, c are any three distinct real numbers, then one of the following must be true: either $a < b < c$, $a < c < b$ or $b < a < c$.

22. Given two points P and Q on a line ℓ, only one coordinate system on ℓ assigns 0 to P and 1 to Q.

23. The union of two halfplanes is a plane.

24. If two lines intersect so that a pair of vertical angles formed are supplementary, then the measure of each of these angles is 90.

25. If A, B, C are points and B is between A and C, then $BA + BC = CA$.

26. The important geometric statement that three noncollinear points determine a plane has also significant meaning for practical applications in the physical world.

27. The "if" part of a theorem in "if ... then ... " form is the hypothesis.

28. There is a unique number which is the measure of the distance between any two points in space relative to a chosen unit-pair.

29. If P, Q, R are three distinct points on a line to which a coordinate system assigns numbers p, q, r respectively and if $q < p < r$ then Q lies between P and R.

30. The interior of an angle contains the interior of all rays between the sides of the angle.

31. If a point is in the interior of two angles of a triangle, then it is in the interior of the triangle.

32. To build a deductive system, it is necessary to accept some statements without proof.

33. Given a line there is one and only one plane containing it.

34. If S is the set of all integers x such that $2 \leq x \leq 5$, and if T is the set of all integers y such that $y = x + 4$ for some x in S, then it is possible to establish a one-to-one correspondence between the elements of S and the elements of T.

35. If A, B, C are points and a, b, c are numbers, there is a unique one-to-one correspondence between the points and the numbers.

36. Protractors are always marked in degree units.

37. Congruences between triangles have the transitive, symmetric and reflexive properties.

38. A statement that something is true if and only if something else is true can be rephrased to form two statements.

39. If two distinct planes m and n intersect, their intersection is an infinite set of points.

40. If $kx > 7$ and $k < 0$, then $x > \frac{7}{k}$.

41. An infinite number of distinct coordinate systems exist on any line.

42. If two distinct points lie in the same halfplane, then the line determined by them does not intersect the edge of that halfplane.

43. If two supplementary angles form a linear pair, then one of the angles must be acute.

44. $\triangle ABC \cong \triangle DEF$ if $\angle A \cong \angle D$, $\overline{AC} \cong \overline{DF}$ and $\overline{BC} \cong \overline{EF}$.

45. The postulates for formal geometry are the same in all geometry books.

46. A statement obtained by logical deduction from postulates and theorems in our formal geometry may be considered as a theorem in our formal geometry.

47. Given two distinct points on a line a coordinate system can be chosen so that the coordinate of one point is zero and the coordinate of the other point is a negative number.

48. If A, B, C are three distinct collinear points then B is between A and C if $AB + BC = AC$.

49. An angle is never determined by two segments.

50. Two right triangles may be congruent if only one side and one angle of one are congruent to the corresponding side and angle of the other.

51. Geometry is related to things in the real world even though the points which form geometric objects exist only in our minds.

52. To say a set of collinear points is a line is the same as to say a line is a set of collinear points.

53. The set of integers is a subset of the set of irrational numbers.

54. If A, B, C are collinear points to which a correspondence has assigned the numbers -2, 1, -5, respectively, then the midpoint of $\overline{BC}$ is A .

55. An angle must always be considered as a set of points.

56. One triangle may be congruent to another without being equal to it.

57. In an isosceles triangle the median to the base bisects the vertex angle.

58. Four noncollinear points determine six lines.

59. The properties of order in our text are properties of real numbers.

60. If the distance between two points relative to one unit-pair is twice as large as the measure between the same two points relative to a second unit-pair, then the measure of the distance between the points of the second unit-pair relative to the first unit-pair is 2 .

61. A point on the edge of a halfplane belongs to that halfplane.

62. If point Q is in the exterior of $\angle ABC$, then Q and C are on the same side of $\overleftrightarrow{AB}$.

63. If $\triangle XYZ \cong \triangle CAB$, then $\angle A \cong \angle X$.

64. A deductive geometry is useless unless all the terms which are used are defined.

65. If a statement is true, its converse is also true.

66. There is a point on a number line which cannot be matched with a real number.

67. The graph of the solution set of the inequality $3x + 3 \geq 2x + 1$ is a ray.

68. If in some ray coordinate system the numbers 20, 30, 40 are assigned, respectively, to rays $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$, then $m\angle AVB + m\angle CVB = m\angle AVC$.

69. The S.S.S. Postulate is just as usable in deductive proof as if it had been a S.S.S. Theorem.

70. If the medians of a triangle are the same as the angle bisectors and also the same as the segments from the vertices perpendicular to the opposite side, then the triangle is equilateral.

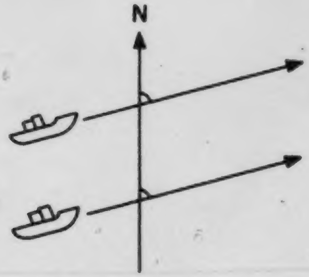
Chapter 6

PARALLELISM

6-1. Introduction.

Our study in the last chapter led us from an idea in the physical world, namely the idea of "same size and shape," to a concept in formal geometry, the concept of congruence. In this chapter we examine another idea in the physical world, namely parallelism, and then develop the formal mathematical concept of two lines being parallel.

What does "parallel" suggest to you? Perhaps the "lines" between lanes on a running track, or the cracks between boards in a floor, or the strings on a harp? What properties of parallel lines in the physical world seem to you to be important? If two ships are sailing parallel courses close to one another as pictured in the diagram, how do the angles indicated in the figure compare?



The congruence of the angles is an example of an idea from the physical world which suggests a property of abstract parallel lines.

Another property of two parallel lines is that they are everywhere the same distance apart. A pair of railroad tracks must accommodate wheels which are fixed so that the distance between them cannot change as they roll down the track. These and other properties of parallelism in the physical world have their abstract counterparts in the theory of parallel lines.

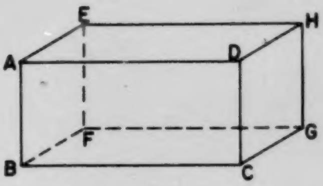
Our next objective is a mathematical treatment of parallelism, an important concept derived from our experience with real objects.

6-2. Definitions.

DEFINITION. Two coplanar lines (whether distinct or not) whose intersection is not a set consisting of a single point are called parallel lines, and each is said to be parallel to the other.

Notation. If p and q are lines, we denote the statement that "p is parallel to q" by: $p \parallel q$.

Notice that there are two requirements in the definition of parallel lines: (1) the lines are coplanar, and (2) the lines, if they are distinct, do not intersect. If p and q are any lines, there are four distinct possibilities: (1) there is no plane which contains both p and q , in which case p and q are skew lines; (2) p and q are coplanar and intersect in one point; (3) p and q are coplanar and do not intersect, in which case they are parallel; (4) p and q are the same line, in which case they are parallel.



This is a drawing of a rectangular box.
Name two edges which lie on parallel lines.
Name two edges which lie on intersecting lines.
Name two edges which lie on skew lines.

Let p and q be two lines which lie in a plane $\mathcal{E}$. Another line in $\mathcal{E}$ may intersect both p and q , or it may not. If it intersects both p and q , it may intersect them in distinct points, or all three lines may have one point in common.

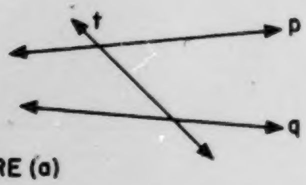


FIGURE (a)

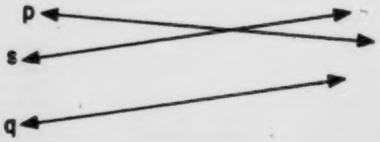


FIGURE (b)

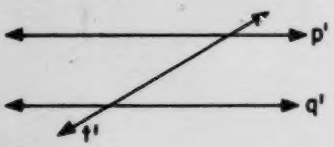


FIGURE (c)

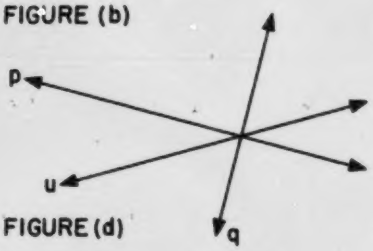


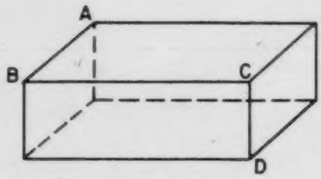
FIGURE (d)

In Figure (a), the line t intersects both p and q ; in Figure (b) the line s intersects p but not q . In Figure (d), the line u intersects both p and q , and the point of intersection of u and p is the same as the point of intersection of u and q . In Figure (a), the intersection points of t with p and with q are distinct; we call the line t a transversal of the lines p and q . Likewise, in Figure (c), the line t' intersects the union of p' and q' in exactly two points; again we call t' a transversal of p' and q' . Try to formulate a definition for a transversal.

DEFINITION. A transversal of two distinct coplanar lines is a line which intersects their union in exactly two points.

In Figure (b), line s is not a transversal of lines p and q . Why? In Figure (d), line u is not a transversal of p and q . Why?

In the diagram below, the points A, B, C, D are four of the vertices of a rectangular box. The line $\overleftrightarrow{BC}$ intersects the union of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ in two points; nevertheless $\overleftrightarrow{BC}$ is not a transversal of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. Why?



If one line is a transversal of two other lines, explain why all three lines are coplanar.

A transversal of two lines forms with the lines eight angles. Certain pairs of these angles have special importance and we introduce names to describe them. Figure (e) shows a transversal t of two lines p , q . Angles $\angle\alpha$, $\angle\beta$, $\angle\theta$, $\angle\phi$ are considered to be interior angles. We call $\angle\alpha$ and $\angle\phi$ consecutive interior angles. Figure (f) shows the same pair of angles. We notice that $\angle\alpha$ and $\angle\phi$ contain a common segment and that their interiors intersect. Another pair of consecutive interior angles in Figure (e) are $\angle\theta$ and $\angle\beta$. We say that $\angle\alpha$ and $\angle\beta$ in Figure (e) are a pair of alternate interior angles. The same pair of angles is shown in Figure (g). We notice that $\angle\alpha$ and $\angle\beta$ contain a common segment and that their interiors do not intersect. There is another pair of alternate interior angles in Figure (e); which two angles do you think they are?

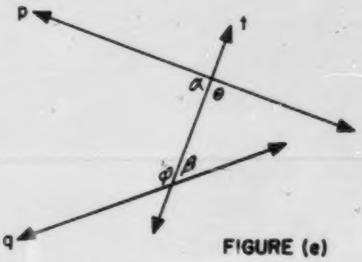


FIGURE (e)

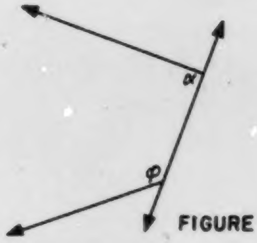


FIGURE (f)

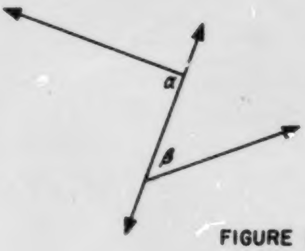


FIGURE (g)

In Figure (h), $\angle\beta$ and $\angle\gamma$ are vertical angles and $\angle\alpha$ and $\angle\beta$ are alternate interior angles; we say that $\angle\alpha$ and $\angle\gamma$ are a pair of corresponding angles. What is the intersection of $\angle\alpha$ and $\angle\gamma$? What is the intersection of the interior of $\angle\alpha$ and the interior of $\angle\gamma$? We now state the definitions of the notions introduced in this paragraph.

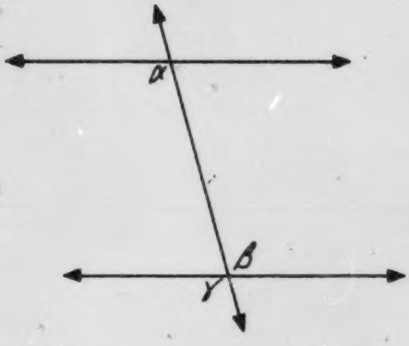


FIGURE (h)

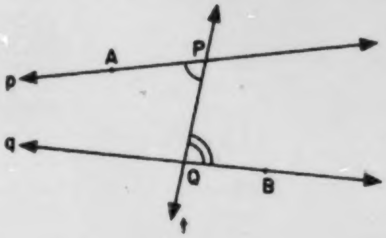
DEFINITION. Two coplanar angles are called a pair of alternate interior angles if and only if the intersection of the angles is a segment and the interiors of the angles do not intersect.

DEFINITION. Two angles are called a pair of consecutive interior angles if and only if the intersection of the angles is a segment and the interiors of the angles intersect.

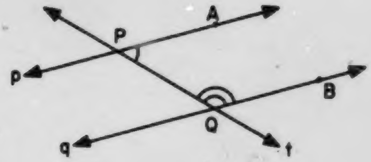
DEFINITION. Two angles with distinct vertices are called a pair of corresponding angles if and only if the intersection of the angles is a ray and the interiors of the angles intersect.

Let p and q be two distinct coplanar lines and let t be a transversal of p and q, intersecting them in P and Q, respectively. Let $\overrightarrow{PA}$ and $\overrightarrow{QB}$ be rays contained in p and q, respectively.

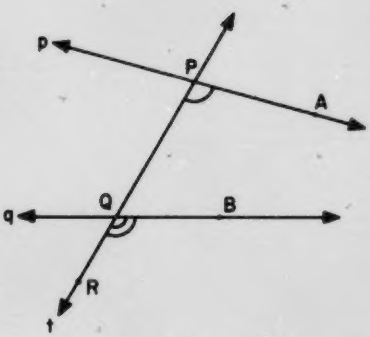
(1) If A and B are on opposite sides of t, then $\angle APQ$ and $\angle BQP$ are a pair of alternate interior angles.



(2) If A and B are on the same side of t, then $\angle APQ$ and $\angle BQP$ are a pair of consecutive interior angles.



(3) If A and B are on the same side of t, then $\angle APQ$ and $\angle BQR$, where $\overrightarrow{QR}$ is opposite to $\overrightarrow{QP}$, are a pair of corresponding angles.



Sometimes we speak of a pair of alternate interior angles, or a pair of consecutive interior angles, or a pair of corresponding angles, "determined by" two lines and a transversal of them. In every such case, the segment or the ray which is the intersection of the two angles is contained in the transversal of the lines.

Problem Set 6-2

1. The diagram shows a transversal t of two coplanar lines p and q.

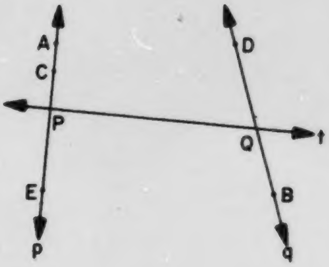
   Fill each blank with one word or two words or three capital letters:

   (a) $\angle EPQ$ and $\angle DQP$ are a pair of ______ angles.

   (b) $\angle APQ$ and $\angle DQP$ are a pair of ______ angles.

   (c) $\angle BQP$ and $\angle$ ______ are consecutive interior angles.

   (d) $\angle CPQ$ and $\angle$ ______ are alternate interior angles.



2. The diagram shows a transversal t of two coplanar lines p and q .

   Fill each blank with one word or two words or a single letter:

   (a) ∠b and ∠d are a pair of ______ angles.

   (b) ∠d and ∠v are a pair of ______ angles.

   (c) ∠b and ∠v are a pair of ______ angles.

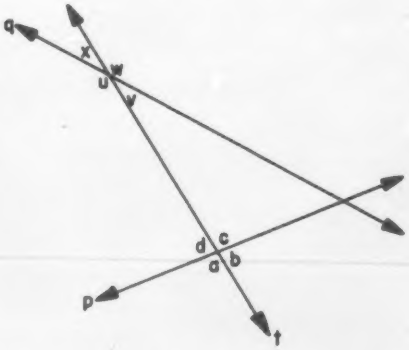
   (d) ∠x and ∠d are a pair of ______ angles.

   (e) ∠w and ∠ ______ are corresponding angles.

   (f) ∠a and ∠ ______ are corresponding angles.

   (g) ∠a and ∠c are ______ angles.

   (h) ∠u and ∠v are ______ angles.



   (i) There are ______ (how many) pairs of alternate interior angles determined by p , q , and the transversal t .

   (j) There are ______ (how many) pairs of consecutive interior angles determined by p , q , and the transversal t .

   (k) There are ______ (how many) pairs of corresponding angles determined by p , q , and the transversal t .

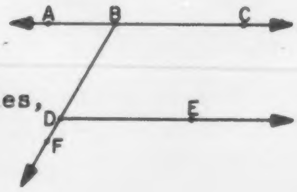
Each of Problems 3-15 is accompanied by a plane figure on the right.

3. Consider only the angles that can be named in terms of three points labeled in the diagram. Identify the following:

   (a) a pair of corresponding angles,

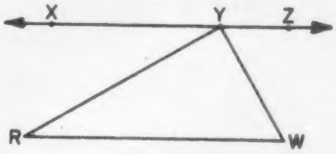
   (b) a pair of alternate interior angles,

   (c) a pair of consecutive interior angles.



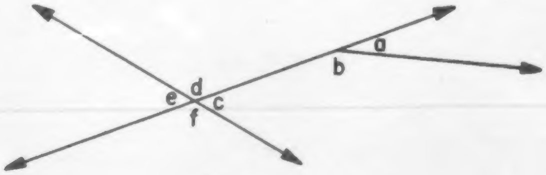
4. Consider only the angles that can be named in terms of three points labeled in the diagram. Identify, if any, all pairs of:

   (a) alternate interior angles,
   (b) consecutive interior angles,
   (c) corresponding angles.



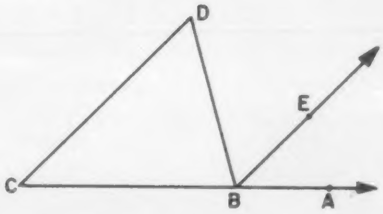
5. What name can be given to each of the following pairs of angles?

   (a) $\angle a$ and $\angle c$,
   (b) $\angle b$ and $\angle c$,
   (c) $\angle b$ and $\angle d$,
   (d) $\angle b$ and $\angle f$.



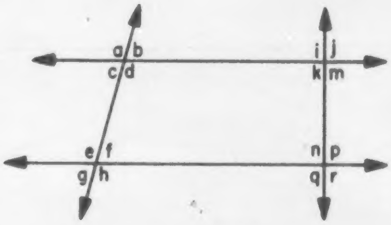
6. Consider only the angles that can be named in terms of three points labeled in the diagram. Identify all pairs of:

   (a) alternate interior angles,
   (b) consecutive interior angles,
   (c) corresponding angles.



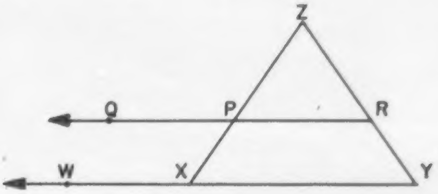
7. Consider only angles that are labeled by a single letter in the diagram. Identify all pairs of:

   (a) corresponding angles,
   (b) alternate interior angles,
   (c) consecutive interior angles.



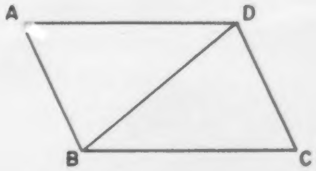
8. Consider only the angles that can be named in terms of three points labeled in the diagram. Identify all pairs of corresponding angles, all pairs of alternate interior angles, and all pairs of consecutive interior angles which are determined by two lines and a transversal of them, if the transversal is:

(a) $\overleftrightarrow{XZ}$,
(b) $\overleftrightarrow{RQ}$,
(c) $\overleftrightarrow{YW}$.



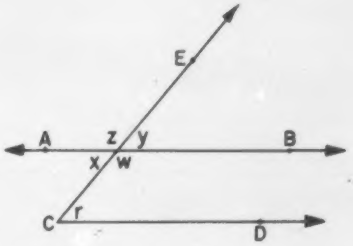
9. Identify the transversal and the two lines intersected by the transversal which determine:

(a) ∠ABD and ∠CDB as a pair of alternate interior angles;
(b) ∠ADB and ∠CBD as a pair of alternate interior angles;
(c) ∠BAD and ∠CDA as a pair of consecutive interior angles.
(d) ∠ADC and ∠BCD as a pair of consecutive interior angles.

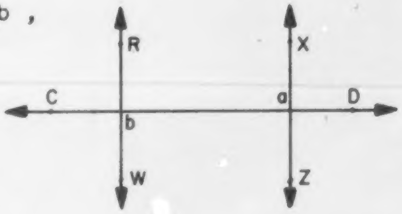


*10. If m ∠r = 50 and m ∠r = m ∠x , find:

(a) m ∠y ;
(b) m ∠w .

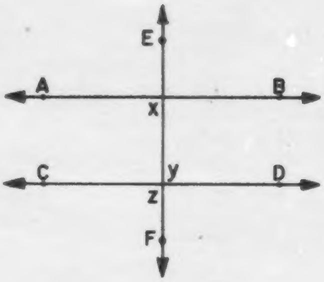


*11. If $\overleftrightarrow{XZ} \perp \overleftrightarrow{CD}$ and m ∠a = m ∠b , is $\overleftrightarrow{RW} \perp \overleftrightarrow{CD}$? Why?

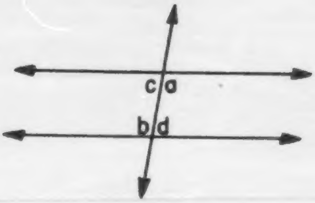


*12. If $\overleftrightarrow{AB} \perp \overleftrightarrow{EF}$ and $\overleftrightarrow{CD} \perp \overleftrightarrow{EF}$, determine whether each of the following is true and explain why:

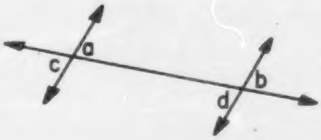
(a) $m\angle x = m\angle y$;
(b) $m\angle x = m\angle z$.



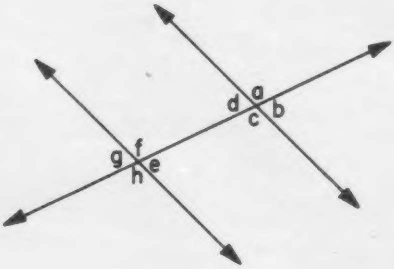
*13. If $m\angle a = 100$ and $m\angle a = m\angle b$, find $m\angle c$ and $m\angle d$.



*14. If $m\angle a = m\angle b$ and $m\angle a = 70$, find $m\angle c$ and $m\angle d$.



*15. If $m\angle a = 110$ and $m\angle d = m\angle e$, find the measure of each of the other angles which are labeled in the diagram by a single letter.



*16. Prove: Let two lines and a transversal of the lines be given. Consider pairs of alternate interior angles, pairs of consecutive interior angles, and pairs of corresponding angles, all determined by the two lines and the transversal. If one pair of alternate interior angles are congruent, then:

(a) the other pair of alternate interior angles are congruent;

(b) each pair of consecutive interior angles are supplementary; and

(c) each pair of corresponding angles are congruent.

*17. Prove: Let two lines and a transversal of the lines be given. Consider pairs of alternate interior angles, pairs of consecutive interior angles, and pairs of corresponding angles, all determined by the two lines and the transversal. If one pair of consecutive interior angles are supplementary, then:

(a) a pair of alternate interior angles are congruent;
(b) the other pair of consecutive interior angles are supplementary; and
(c) each pair of corresponding angles are congruent.

Hint: Use the results of Problem 16 to prove (b) and (c).

*18. Prove: Let two lines and a transversal of the lines be given. Consider pairs of alternate interior angles, pairs of consecutive interior angles, and pairs of corresponding angles, all determined by the two lines and the transversal. If one pair of corresponding angles are congruent, then:

(a) a pair of alternate interior angles are congruent;
(b) each of the other pairs of corresponding angles are congruent; and
(c) each pair of consecutive interior angles are supplementary.

Hint: Use the results of Problem 16 to prove (b) and (c).

6-3. Indirect Method of Proof.

On several occasions in earlier chapters we have used "indirect reasoning." In Section 1-5 the problem of the childless couple was solved by the indirect method. Theorems 2-4 and 2-8 were proved "indirectly." Other applications of this valuable method have been made in other chapters. We wish to analyze this method in more detail.

As a first example, let us examine again the plan for the proof of Theorem 2-4. (You should reread the theorem and its proof at this time.) We are given two distinct lines. By hypothesis, they have at least one point in common. Our task is to show they have exactly one point in common. There are certainly only two cases which appear possible: either the lines have exactly one common point or else they do not. We examine the second alternative and are able, after a few steps, to reject it as a possibility. This rejection leaves only one case as possible (namely, that the lines intersect in exactly one point) and this statement is the desired conclusion.

In brief, indirect reasoning involves the following: list all cases as possibilities and then reject all cases except the desired one. A case is rejected if it can be shown to lead to a contradiction. In greater detail, the steps in a proof by the indirect method (after the identification of hypothesis and conclusion) may be listed as follows:

(1) List all the "possibilities," of which the conclusion is one.

(2) Show that each of these "possibilities," excluding the conclusion, either itself contradicts, or else leads to a statement that contradicts, the hypothesis or a postulate or a previous theorem.

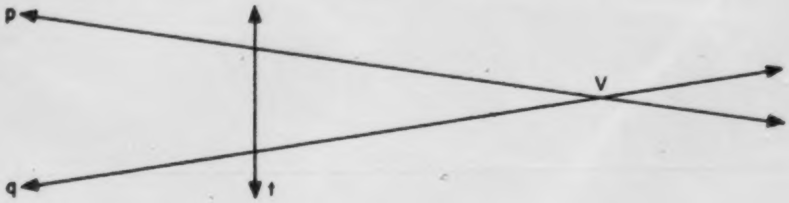
(3) If you are able to reject every "possibility," excluding the conclusion, then the conclusion is actually the only possibility, and the theorem is proved.

Very often in practice, we consider only two cases, namely the desired conclusion and its denial. We then show that the denial of the conclusion leads to a contradiction. When this has been done, we are left with only one possibility, namely the desired conclusion.

The proof of the next theorem is an example of the indirect method. We are given two coplanar lines satisfying a certain requirement. We wish to show that the lines are parallel. The two cases we consider are: (1) the lines are parallel, and (2) the lines are not parallel. These are, respectively, the desired conclusion and its denial. We examine the second case and show that it leads to a contradiction of a previous theorem. Thus we are obliged to accept the conditional which is the statement of the theorem under discussion. With the plan of a proof in mind, we now state and prove the theorem.

THEOREM 6-1. Let two distinct coplanar lines be given. If a transversal of the lines is perpendicular to each of them, then the lines are parallel.

Proof: Let the two given lines be p and q, and let the given transversal be t. Either p and q are parallel, or they are not. Suppose p and q are not parallel. Then they intersect in exactly one point, say V. By the definition of a transversal, V is not on t.



Now each of the two lines p and q contains V; and each is, by hypothesis, perpendicular to t. Since p and q are distinct, the preceding statement is a contradiction of the uniqueness part of Theorem 5-11. We reject the alternative that p and q are not parallel. Hence p and q are parallel.

Since the indirect method often requires that we formulate the denial of a statement, we should look at several examples.

Statement	Denial of the Statement
$\angle A$ is a right angle.	$\angle A$ is not a right angle.
x is greater than 5.	x is not greater than 5.
Richard is not here.	Richard is here.
$a \neq y$.	$a = y$.
$AB = CD$	$AB \neq CD$.
The two lines intersect.	The two lines do not intersect.
Set A is different from Set B.	Set A is the same as Set B.

Most of our theorems are conditional statements, that is, statements in the "If ---, then ---" form. More specifically, we have: "If hypothesis, then conclusion." One type of indirect reasoning is to consider the case which is the denial of the conclusion and show that this assumption leads to a contradiction of the hypothesis. In effect, the procedure is to show:

If (denial of conclusion), then (denial of hypothesis). When this task is accomplished, we consider the given conditional statement as proved. The remarks lead us to consider the "contrapositive" of a conditional statement.

Consider a conditional statement of the type "If p, then q," where p and q are statements which are respectively the hypothesis and the conclusion. The contrapositive of the given conditional statement is: "If denial of q, then denial of p."

Our discussion of indirect reasoning suggests that a conditional can be proved by establishing its contrapositive. In each of the following examples, check the statement and its contrapositive to see whether they are both true or both false, or whether one is true and the other false.

Example 1.

Statement: If $\overline{AB} \cong \overline{CD}$, then $AB = CD$.

Contrapositive: If $AB \neq CD$, then $\overline{AB}$ is not congruent to $\overline{CD}$.

Example 2.

Statement: If I live in Detroit, then I live in California.

Contrapositive: If I do not live in California, then I do not live in Detroit.

Example 3.

Statement: If two angles are a linear pair of angles, then they are supplementary.

Contrapositive: If two angles are not supplementary, then they are not a linear pair of angles.

Your check should indicate that in each case the conditional statement and its contrapositive agree in truth value. We may call such a pair of statements logically equivalent.

In Section 5-9 we saw that a conditional and its converse do not need to be logically equivalent, for one of them may be true and the other false. However it appears that a conditional and its contrapositive are logically equivalent. A diagram may help us to understand these ideas. In the figure consider the smaller circular region A which is contained in the larger circular region B . A conditional statement describing this situation is:

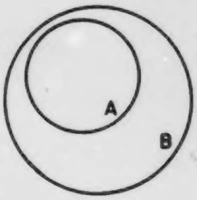
If a point is in A , then it is in B .

The converse of the given conditional is:

If a point is in B , then it is in A .

The contrapositive of the given conditional is:

If a point is not in B , then it is not in A .



The conditional and the contrapositive are both true, but the converse is not.

Furtnermore, the contrapositive of

"If denial of q , then denial of p"

is the statement, "If p , then q ." This means that the contrapositive of a contrapositive is the original conditional. Therefore, if a contrapositive is true, then the original statement is also true.

Property of the Contrapositive

A conditional statement and its contrapositive are logically equivalent.

This property can be valuable to us when we wish to prove a theorem that has the "If ---, then ---" form. We can either prove the theorem or its contrapositive, thereby proving both. Sometimes it is easier to prove the contrapositive.

The proof of the main theorem in the next section will illustrate the use of the Property of the Contrapositive.

Problem Set 6-3

1. (a) Write the converse and the contrapositive of Theorem 6-1.
   (b) Which of these statements can be accepted as true at this time? Why?
2. (a) Write the converse and the contrapositive of the statement:

   "If two sides of a triangle are congruent, the angles opposite these sides are congruent."

   (b) Which of these statements can be accepted as true at this time? Why?

*3. Write the contrapositive of the following statement:

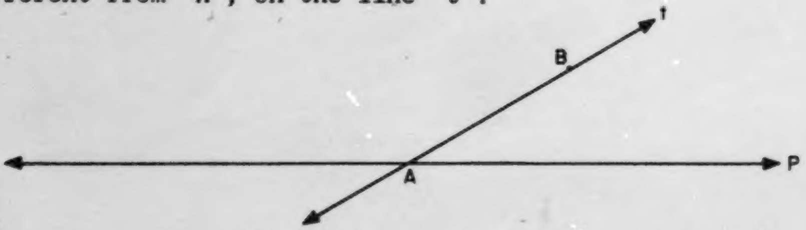
"If two distinct lines are parallel, then any two alternate interior angles determined by a transversal of the lines are congruent."

6-4. Parallel Line Theorems.

The following experiment (in physical world geometry) leads us to connect two relations, the relation of congruence between two alternate interior angles determined by two lines and a transversal, and the relation of parallelism between the two lines.

Experiment

On a large sheet of paper draw a figure showing two lines p and t intersecting at a point A . Mark a point B , different from A , on the line t .

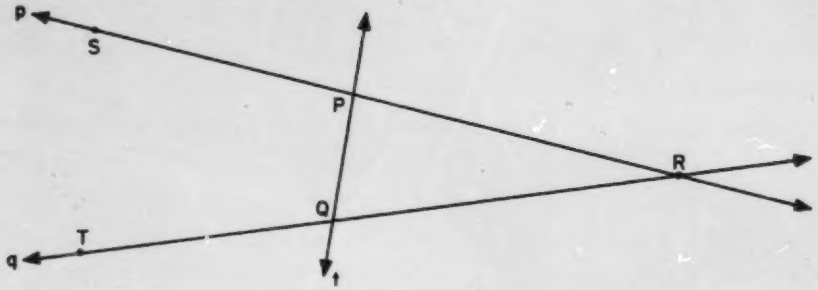


Using your protractor and a ruler, draw a line $\overleftrightarrow{BD}$ such that two alternate interior angles determined by p , $\overleftrightarrow{BD}$, and the transversal t will have equal measures.

Does it appear that $\overleftrightarrow{BD}$ and p are parallel? Making two alternate interior angles congruent seems to make the lines parallel. Do you think that in the mathematical world "alternate interior angles are congruent" implies "lines are parallel"? We proceed to state and prove this formally.

THEOREM 6-2. Let two distinct coplanar lines be given. If two alternate interior angles determined by a transversal of the lines are congruent, then the lines are parallel.

Proof: Let p and q be the given distinct lines. Let the given transversal t intersect the lines p and q in the respective points P and Q . Either p and q are parallel or they are not. We use the indirect method of proof.

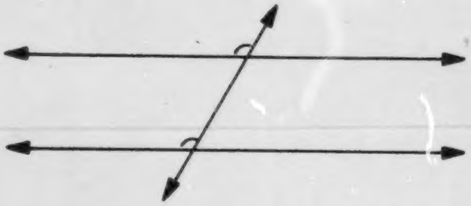


Statements	Reasons
1. p and q intersect at a point, say R .	1. Denial of desired conclusion.
2. P, Q, R are not collinear.	2. Definition of transversal.
3. There are points S and T on p and q such that $\angle QPS$ and $\angle PQT$ are exterior angles of $\triangle PQR$.	3. Definition of exterior angle of a triangle.
4. m $\angle QPS >$ m $\angle PQR$, and m $\angle PQT >$ m $\angle QPR$.	4. Exterior Angle Theorem (Theorem 5-10).
5. m $\angle QPS =$ m $\angle PQR$, and m $\angle PQT =$ m $\angle QPR$.	5. By hypothesis, the alternate interior angles in one pair are congruent; the angles of the other pair are congruent, since they are supplements of congruent angles.
6. p $\parallel$ q .	6. Denial of conclusion contradicts the hypothesis (Steps 4, 5).

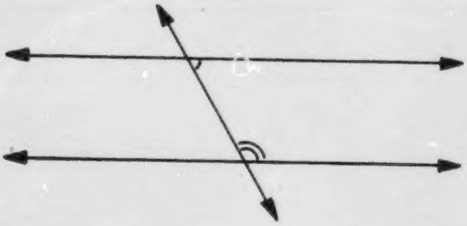
The contrapositive of Theorem 6-2 may be stated as follows: "Let two distinct coplanar lines be given. If the lines are not parallel, then no two alternate interior angles determined by a transversal of the lines are congruent." Notice that the proof we gave for Theorem 6-2 is, in effect, a proof of the contrapositive. We supposed that the lines are not parallel and showed that in each pair of alternate interior angles, one of the angles has greater measure than the other. On the basis of our discussion in the preceding section, we then considered Theorem 6-2 as proved.

The proofs of the following two corollaries are left as problems.

Corollary 6-2-1. Let two coplanar lines be given. If two corresponding angles determined by a transversal of the lines are congruent, then the lines are parallel.

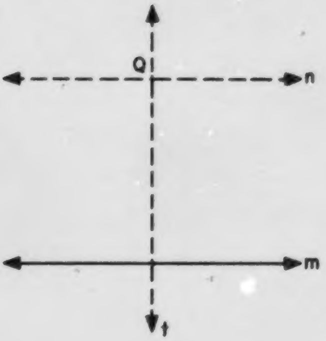


Corollary 6-2-2. Let two coplanar lines be given. If two consecutive interior angles determined by a transversal of the lines are supplementary, then the lines are parallel.



THEOREM 6-3. Let a line and a point not on the line be given. In the plane determined by the line and the point, there is a line which contains the given point and is parallel to the given line.

Proof: Let the given line and the given point be denoted by m and Q , respectively. Let $\mathcal{E}$ be the plane determined by m and Q . Since Q is not on m , there is in $\mathcal{E}$, by Theorem 5-11, a line t containing Q and perpendicular to m . By Theorem 4-21, there is in $\mathcal{E}$ a line n containing Q and perpendicular to t . Since t is a transversal of the distinct lines m and n in plane $\mathcal{E}$, the lines m and n are parallel, by Theorem 6-1.



Summary.

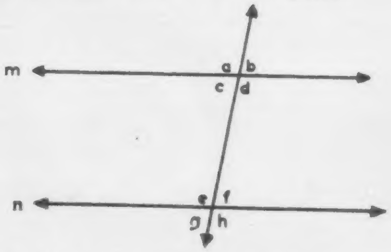
Two distinct coplanar lines are parallel if any one of the following conditions is satisfied:

1. The lines do not intersect. (Definition)
2. A pair of alternate interior angles determined by a transversal of the lines are congruent. (Theorem 6-2)
3. A pair of corresponding angles determined by a transversal of the lines are congruent. (Corollary 6-2-1)
4. A pair of consecutive interior angles determined by a transversal of the lines are supplementary. (Corollary 6-2-2)
5. Each of the lines is perpendicular to a transversal of the two lines. (Theorem 6-1)

Problem Set 6-4

1. Write a proof for Corollary 6-2-1.
2. Write a proof for Corollary 6-2-2.
3. Write a proof for Theorem 6-1, using Theorem 6-2 and the direct method.

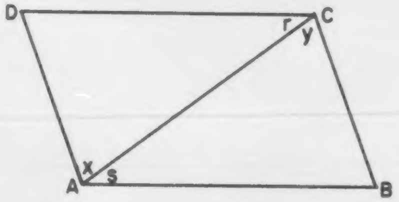
Each of the problems in 4-18 is accompanied by a plane figure on the right.



4. Why is m || n if:

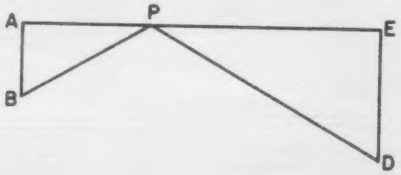
   (a) $m\angle a = 100$, $m\angle e = 100$
   (b) $m\angle c = 80$, $m\angle f = 80$
   (c) $m\angle d = 100$, $m\angle h = 100$
   (d) $m\angle c = 80$, $m\angle e = 100$

In Problems 5 and 6 name the distinct lines which are parallel as a result of the given conditions. State the reason for your answer.

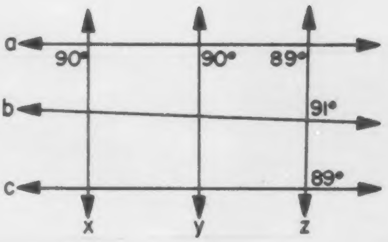


5. (a) If $\angle x \cong \angle y$.
   (b) If $\angle r \cong \angle s$.
   (c) If $\angle ADC$ and $\angle BCD$ are supplementary.
   (d) If $\angle BAD$ and $\angle CDA$ are supplementary.

6. If $\overline{AB} \perp \overline{AE}$ and $\overline{DE} \perp \overline{AE}$.

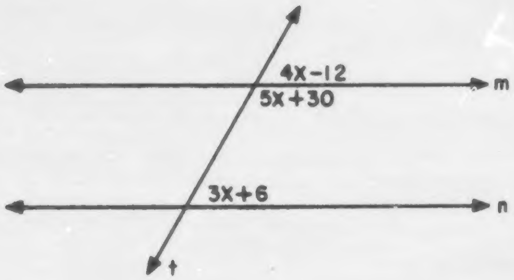


7. If the measures of the angles are as indicated in the drawing, what distinct lines are parallel? Why?

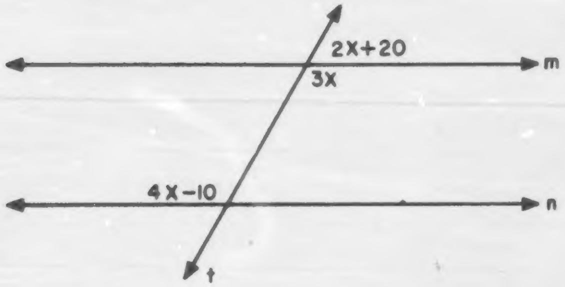


In Problems 8 and 9 find the number x and determine whether $m \parallel n$. Supply a reason for your answer.

8.

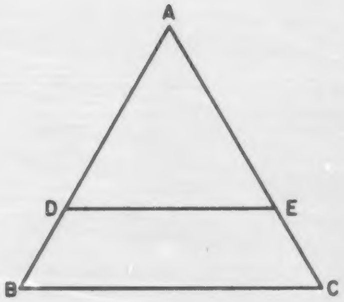


9.



10. Hypothesis: D is between A and B; E is between A and C. $\overline{AD} \cong \overline{AE}$; $\overline{AB} \cong \overline{AC}$; $\angle C \cong \angle ADE$.

Prove: $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$.



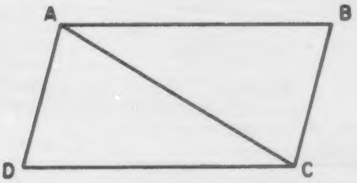
11. Hypothesis:

$AB = CD$

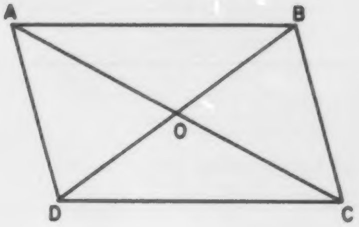
$AD = CB$

Prove: $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$,

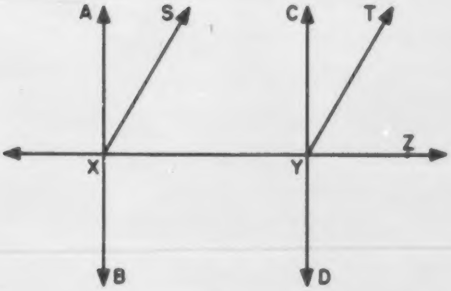
$\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$.



12. Hypothesis:
$\overline{AC}$ and $\overline{BD}$ bisect each other.
Prove: $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$, $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$.



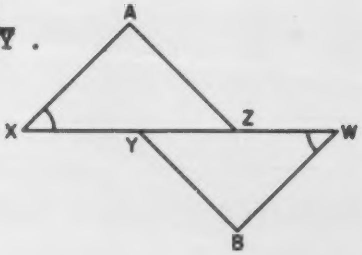
13. Hypothesis:
$\overleftrightarrow{AB} \perp \overleftrightarrow{XY}$; $\overleftrightarrow{CD} \perp \overleftrightarrow{XY}$; $\angle AXS \cong \angle CYT$; $\overrightarrow{XS}$ is between $\overrightarrow{XA}$ and $\overrightarrow{XZ}$; $\overrightarrow{YT}$ is between $\overrightarrow{YC}$ and $\overrightarrow{YZ}$.
Prove: $\overleftrightarrow{XS} \parallel \overleftrightarrow{YT}$.



14. Hypothesis:
Points X, Y, Z, W are collinear in that order. Triangles XAZ and WBY are isosceles triangles with respective bases $\overline{XZ}$ and $\overline{WY}$. $\angle X \cong \angle W$.

(a) Prove: $\overleftrightarrow{AZ} \parallel \overleftrightarrow{BY}$.

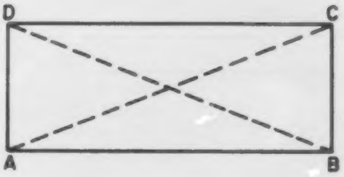
(b) Would the proof apply if $\triangle XAZ$ and $\triangle WBY$ are not coplanar?



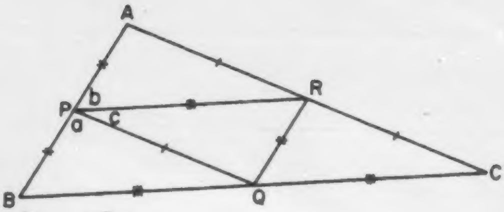
15. Hypothesis:
$m\angle DAB = m\angle CBA = 90$, and $AD = CB$.

Prove: $m\angle ADC = m\angle BCD$.

Is it true that $m\angle ADC = 90$? Why?



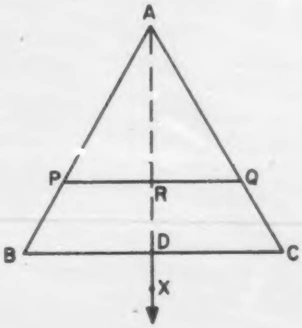
16. Given the figure with
$AR = RC = PQ$,
$AP = PB = RQ$,
$BQ = QC = PR$.



Prove: $m\angle A + m\angle B + m\angle C = 180$.

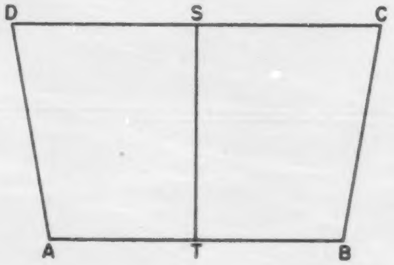
Hint: Prove $m\angle a = m\angle A$, $m\angle b = m\angle B$, $m\angle c = m\angle C$.

17. Hypothesis:
P is between A and B , Q is between A and C , $AB = AC$, $AP = AQ$. $\overrightarrow{AX}$ is the bisector of $\angle A$ and intersects $\overline{PQ}$ at R which is between P and Q and $\overline{BC}$ at D which is between B and C .



Prove: $\overleftrightarrow{PQ} \parallel \overleftrightarrow{BC}$.

18. Hypothesis:
The figure with $\angle A \cong \angle B$, $AD = BC$, T is midpoint of $\overline{AB}$, S is midpoint of $\overline{DC}$.



Prove: $\overline{ST} \perp \overline{DC}$.
$\overline{ST} \perp \overline{AB}$.
$\overleftrightarrow{DC} \parallel \overleftrightarrow{AB}$.

*19. Write the contrapositive of each of the corollaries of Theorem 6-2. Can we accept these statements as true at this time? Why?

*20. Write the converse of Theorem 6-2 and the converse of each of the corollaries of Theorem 6-2. Can we accept these statements as true at this time?

6-5. The Parallel Postulate.

Theorem 6-3 states that there is at least one line which is parallel to a given line and contains a given point not on the given line. At this time it seems natural to try to prove that there is at most one such line. Astonishing as it may appear, this cannot be done on the basis of the postulates that we have stated thus far. We accept the statement as a new postulate and will refer to it as the Parallel Postulate.

> *Postulate 22 (the Parallel Postulate).* There is at most one line parallel to a given line and containing a given point not on the given line.

Together, Theorem 6-3 and Postulate 22 tell us that, if a line and a point not on the line are given, there is exactly one line containing the point and parallel to the given line. Of course, if a line and a point *on* the line are given, then there is exactly one line containing the point and parallel to the given line, namely the given line itself.

It is interesting to note that over a period of several centuries, some very clever people tried to prove this postulate. None of them, however, was able to find a proof. Finally, in the last century, it was discovered that no such proof is possible. The fact is that there are some mathematical systems that are almost like the geometry that we are studying, but not quite. In these mathematical systems, nearly all of the postulates of ordinary geometry are satisfied, but the Parallel Postulate is not satisfied. These "non-Euclidean geometries" may seem strange, and in fact they are. For example, in these geometries there is no such thing as a square. Not only do these geometries lead to interesting mathematical theories, but they also have important applications in physical situations involving either very great (interstellar) or very small (subatomic) distances.

The three mathematicians most frequently associated with the early development of non-Euclidean geometry are a

Hungarian named Bolyai, a Russian named Lobachevski, and a German named Riemann. Although Bolyai and Lobachevski worked independently of each other, each started with the postulate that more than one line can be parallel to a given line through a point not on the line, whereas Riemann assumed that no line can be parallel to a given line. Sometimes these non-Euclidean geometries are referred to as Lobachevskian or hyperbolic geometry, and Riemannian or elliptical geometry. Moreover, it should be noted that Riemann replaced several of Euclid's postulates, including the Parallel Postulate, while Lobachevski and Bolyai modified only the Parallel Postulate.

You should also be familiar with the name of Karl Friedrich Gauss, who (besides making other major contributions to mathematics) actually developed hyperbolic geometry before Bolyai or Lobachevski, but who never published his findings on this subject. To Lobachevski goes much credit for developing hyperbolic geometry in considerable detail.

If you would like to learn how these geometries differ and to understand more about the spirit of the movement which led to the development of non-Euclidean geometry, the following references will provide interesting material for your study.

Eves and Newsom, An Introduction to the Foundations and Fundamental Concepts of Mathematics, Holt-Rinehart-Winston, New York, 1958.

Lieber, Non-Euclidean Geometry, or, Three Moons in Mathesis, Academy Press, New York, 1931.

Richardson, Fundamentals of Mathematics, Macmillan, New York, 1958.

Wolfe, Introduction to Non-Euclidean Geometry, Holt-Rinehart-Winston, New York, 1945.

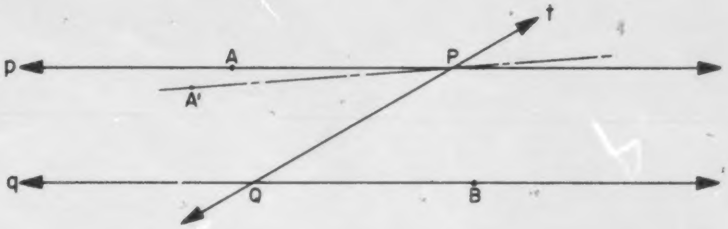
6-6. Additional Parallel Line Theorems.

In Section 6-4 we stated a theorem and several corollaries, each of which may be used to establish that two lines are parallel. In this section we state the converse of each of these statements. The interesting feature about the theorems in this section is that each of them depends upon the Parallel Postulate.

The proof of the next theorem illustrates how the Parallel Postulate may be used. We will consider a certain line q and a certain point P not on q. We will be given a line p containing P and parallel to q. We will find a line (which will be called $\overleftrightarrow{A'P}$) which has an important property and which also contains P and is parallel to q. The Parallel Postulate will enable us to say that $\overleftrightarrow{A'P}$ is the same as p. Then p has the important property too. With this plan as a guide, we state and prove the theorem.

THEOREM 6-4. If two distinct lines are parallel, then any two alternate interior angles determined by a transversal of the lines are congruent.

Proof: Let p and q denote the two given parallel lines. Let the transversal t intersect p and q in the respective points P and Q. Let A and B be points on p and q, respectively, such that $\angle QPA$ and $\angle PQB$ are alternate interior angles. We must prove: $\angle QPA \cong \angle PQB$.



By the Protractor Postulate, there is a point A' on the same side of t as A such that $m\angle QPA' = m\angle PQB$. That is, $\angle QPA' \cong \angle PQB$. Since A and B are on opposite sides of t , A' and B are on opposite sides of t . Thus $\angle QPA'$ and $\angle PQB$ are a pair of alternate interior angles determined by the transversal t of the lines $\overleftrightarrow{A'P}$ and q . Since these two angles are congruent, Theorem 6-2 tells us that $\overleftrightarrow{A'P}$ and q are parallel lines. We now apply the Parallel Postulate: since each of the lines $\overleftrightarrow{A'P}$ and p contains P and is parallel to q , they are the same line. Since p contains A and A' on the same side of t , the rays $\overrightarrow{PA'}$ and $\overrightarrow{PA}$ are the same. In other words, $\angle QPA' = \angle QPA$. Hence $\angle QPA \cong \angle PQB$.

With reference to the above diagram, we remark that, when A' is introduced, we do not know whether A' lies on p or not; consequently we do not picture A' as being on the line. Not until the latter part of the proof are we sure that p contains A' .

The following corollaries are easy to prove by the direct proof method. The proofs are left as problems.

Corollary 6-4-1. If two distinct lines are parallel, then any two corresponding angles determined by a transversal of the lines are congruent.

Corollary 6-4-2. If two distinct lines are parallel, then any two consecutive interior angles determined by a transversal of the lines are supplementary.

Corollary 6-4-3. If a transversal is perpendicular to one of two distinct parallel lines, it is perpendicular to the other also.

Summary.

If two distinct lines are parallel, then all of the following conditions are satisfied:

1. The lines are coplanar and do not intersect. (Definition)
2. Any pair of alternate interior angles determined by a transversal of the lines are congruent. (Theorem 6-4)
3. Any pair of corresponding angles determined by a transversal of the lines are congruent. (Corollary 6-4-1)
4. Any pair of consecutive interior angles determined by a transversal of the lines are supplementary. (Corollary 6-4-2)
5. Any transversal perpendicular to one of the lines is also perpendicular to the other. (Corollary 6-4-3)

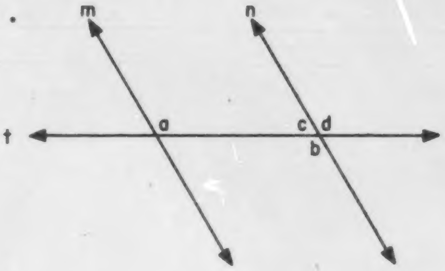
Problem Set 6-6a

1. Prove Corollary 6-4-1.
2. Prove Corollary 6-4-2.
3. Prove Corollary 6-4-3.

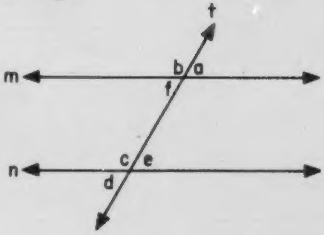
Each of the problems in the remainder of this problem set is accompanied by a plane figure on the right.

4. Given: Transversal t of the parallel lines m and n. Find each of the following and state a reason for your answer.

   (a) $m\angle b$ if $m\angle a = 100$.
   (b) $m\angle c$ if $m\angle a = 100$.
   (c) $m\angle d$ if $m\angle a = 100$.

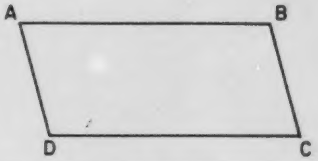


5. Given: $m \parallel n$ with transversal t. Evaluate the measure of the following angles in the order given if $m\angle a = 55$. Give a reason for each answer.

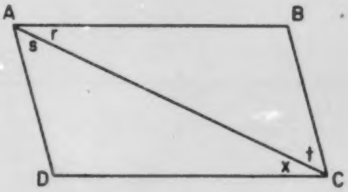


(a) $\angle b$, (d) $\angle e$,
(b) $\angle c$, (e) $\angle f$.
(c) $\angle d$,

6. Name the pairs of supplementary angles if:
(a) $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$, (b) $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$.

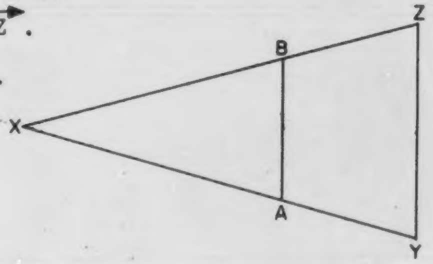


7. Name the pairs of congruent angles if:
(a) $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$, (b) $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$.



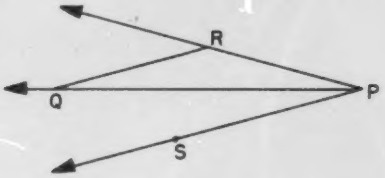
8. Given: $\triangle XYZ$ is isosceles with base $\overleftrightarrow{YZ}$; $\overleftrightarrow{AB} \parallel \overleftrightarrow{YZ}$.

Prove: $\triangle XAB$ is isosceles.



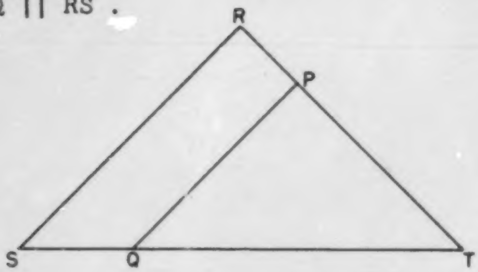
9. Hypothesis: $\overrightarrow{PQ}$ bisects $\angle SPR$, and $\overleftrightarrow{RQ} \parallel \overleftrightarrow{PS}$.

Prove: $\triangle PRQ$ is isosceles.



10. Hypothesis: RT = RS ; $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RS}$.

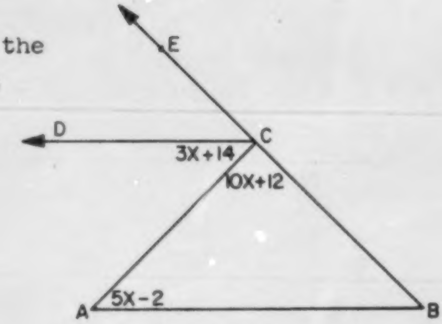
Prove: PQ = PT .



11. Hypothesis: Points B, C, E are collinear; $\overleftrightarrow{DC} \parallel \overleftrightarrow{AB}$.

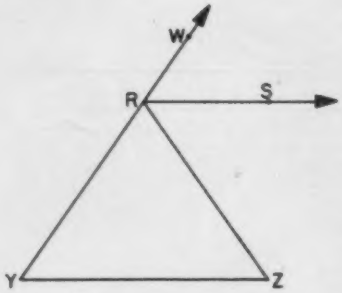
Find the measure of each of the following angles after first determining the number x .

(a) ∠BAC ,
(b) ∠DCA ,
(c) ∠ACB ,
(d) ∠DCE ,
(e) ∠CBA .



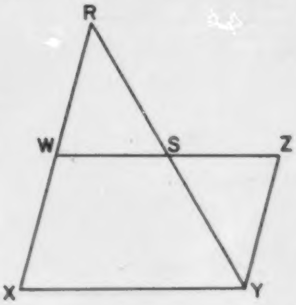
12. Given: $\overrightarrow{RS}$ is the midray of ∠ZRW, $\overleftrightarrow{RS} \parallel \overleftrightarrow{YZ}$.

Prove: △RYZ is isosceles.



13. Hypothesis: In △XRY ,
W is the midpoint of $\overline{XR}$;
S is the midpoint of $\overline{YR}$;
S is the midpoint of $\overline{WZ}$.

Prove: $\overleftrightarrow{YZ} \parallel \overleftrightarrow{XR}$;
$\overline{YZ} \cong \overline{XW}$.



353

14. Hypothesis:

$AB = BC = ED$.

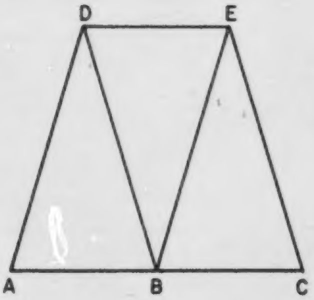
$BE = AD$; $BD = CE$.

Prove:

(a) $\overleftrightarrow{DE} \parallel \overleftrightarrow{AB}$.

(b) $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$.

(c) A, B, and C are collinear.



The following theorem provides another method for establishing that two lines are parallel. It can be proved by the direct or indirect method. We will give the indirect proof. The direct proof is left as a problem.

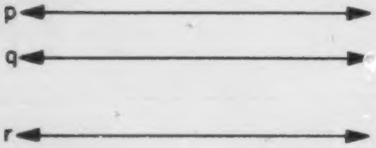
THEOREM 6-5. If each of two coplanar lines is parallel to the same line, they are parallel to each other.

Proof: Let p, q, r be lines such that p and q are coplanar and $p \parallel r$ and $q \parallel r$. Suppose now that all three lines lie in the same plane. There are two possibilities concerning the lines p and q , namely:

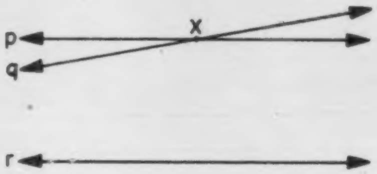
(1) $p \parallel q$,

(2) p intersects q at some point, say X .

The diagrams for each case look like the following:



Case (1)



Case (2)

We will first consider Case (2). If p and q intersect at X , then we have two lines passing through X and each of these lines is parallel to r . Thus we have a contradiction to the Parallel Postulate, which states that only one line is parallel to a given line through a given point. Therefore it is impossible for p and q to intersect at X .

Since Case(2) is impossible, we must accept (1). Therefore $p \parallel q$. The proof is finished for the case that all three lines are coplanar.

Do you think that, in order for the theorem to be valid, it is necessary for all three lines to lie in the same plane? Explain your answer. Give some examples of physical models where the three parallel lines do lie in the same plane; give other examples where the three parallel lines do not lie in the same plane.

Would the mathematical proof for the theorem be different if the three lines do not lie in the same plane? If so, in what ways? If not, explain why.

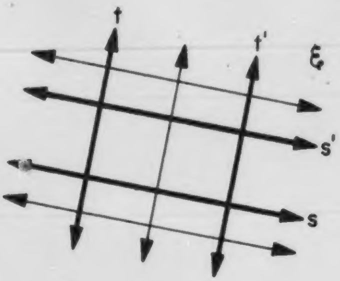
The relationship of parallelism for lines in a given plane has the reflexive, symmetric, and transitive properties. The reflexive and symmetric properties are immediate consequences of the definition of parallel lines, while the transitive property follows from the preceding theorem. Thus parallelism for lines in a given plane has the characteristic properties we have noted before for equality and congruence.

Corollary 6-5-1. If a line lies in the plane of two distinct parallel lines and intersects one of the lines in a single point, then it also intersects the other line in a single point.

The proof is left as a problem.

Corollary 6-5-2. Let $\mathcal{E}$ be a plane, let S be a set of mutually parallel lines in $\mathcal{E}$ (that is, a set of lines in $\mathcal{E}$ such that each line in S is parallel to every other line in S), let T be another set of mutually parallel lines in $\mathcal{E}$. If any one line of S is perpendicular to any one line of T, then every line of S is perpendicular to every line of T.

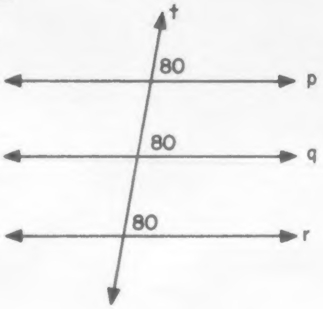
Proof: By hypothesis, there is a line s' of S and there is a line t' of T such that s' is perpendicular to t'. Let s and t be any lines in S and T, respectively. As a first case, suppose that $s \neq s'$ (that is, s and s' are different lines), and suppose that $t \neq t'$.



Since $t' \perp s'$, Corollary 6-5-1 tells us that t' intersects s in a single point and is therefore a transversal of the parallel lines s and s'. By Corollary 6-4-3, t' and s are perpendicular. Then, by Corollary 6-5-1, s is a transversal of the parallel lines t and t'. By Corollary 6-4-3, s and t are perpendicular. In this case, the conclusion is true. The proof for the other cases, namely where $s = s'$ or $t = t'$, is left as a problem.

Problem Set 6-6b

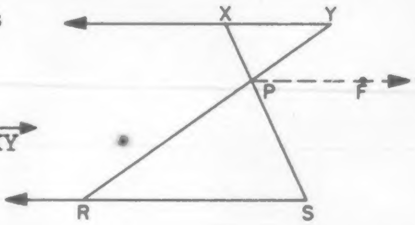
1. In the diagram to the right, the lines p, q, r, and t are coplanar, and the measures of several angles are indicated.



   (a) Is $p \parallel q$? Why?
   (b) Is $q \parallel r$? Why?
   (c) Is $p \parallel r$? Why?

2. In the figure to the right, $\overleftrightarrow{XY} \parallel \overleftrightarrow{RS}$, and $\overline{YR}$ intersects $\overline{XS}$ at the point P . Prove that $m\angle SPY = m\angle S + m\angle Y$.

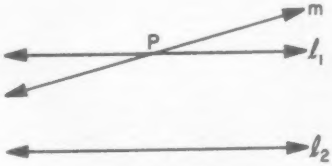
   Suggestion: Consider $\overleftrightarrow{PF} \parallel \overleftrightarrow{XY}$ and prove $\overleftrightarrow{PF} \parallel \overleftrightarrow{RS}$.



3. Review indirect proof as illustrated by the proof of Theorem 6-5.

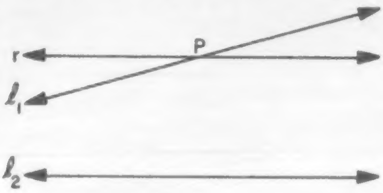
   (a) Give an indirect proof of Corollary 6-5-1 showing a contradiction of the Parallel Postulate.

   Hint: Let m , ℓ_1 , ℓ_2 be three coplanar lines such that $\ell_1 \parallel \ell_2$ and such that m intersects ℓ_1 in some point, say P . If you assume that m does not intersect ℓ_2 , are you led to a contradiction? Why?



   (b) Give an indirect proof of Corollary 6-5-1, showing a contradiction of Theorem 6-5.

(c) Give an indirect proof of the following statement, showing a contradiction of the Parallel Postulate.
In a plane, if a line r intersects only one of two other lines ℓ_1 and ℓ_2 , then the lines ℓ_1 and ℓ_2 intersect.



Hint: If r intersects ℓ_1 say at P , then by hypothesis, r does not intersect ℓ_2 . Show that ℓ_1 and ℓ_2 intersect.

4. Using the direct method, prove Theorem 6-5 for the case in which the three given lines are distinct and coplanar.

5. Complete the proof of Corollary 6-5-2 by examining the cases:

(a) $s = s'$.
(b) $t = t'$.

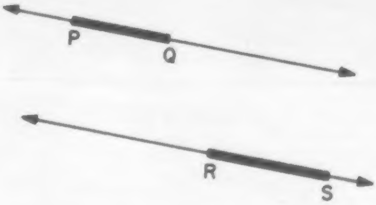
6-7. Parallelism for Segments and Rays.

In this section we introduce a special and important type of convex quadrilateral. Specifically, we consider quadrilaterals whose opposite sides lie on parallel lines.

Our phraseology is conveniently shortened if we extend in a natural manner the notion of parallelism for lines to the case of segments.

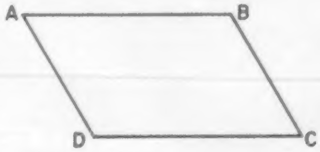
DEFINITION. Two segments which are contained respectively in parallel lines are called parallel segments, and each is said to be parallel to the other.

Let P, Q, R, S be four points. The statement that the segment $\overline{PQ}$ is parallel to the segment $\overline{RS}$ means the same as the statement that the line $\overleftrightarrow{PQ}$ is parallel to the line $\overleftrightarrow{RS}$. We use the same symbol $\|$ to denote that segments are parallel, as for example: $\overline{PQ} \| \overline{RS}$.

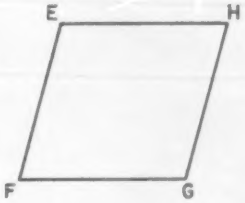


DEFINITION. A *parallelogram* is a quadrilateral each of whose sides is parallel to the side opposite it.

Thus the four sides of a parallelogram constitute two pairs of opposite sides, and each pair of opposite sides is a pair of parallel segments. The parallelogram ABCD has the properties that $\overline{AB} \| \overline{DC}$ and $\overline{AD} \| \overline{BC}$.

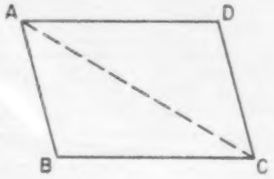


Consider any parallelogram, say the parallelogram EFGH. Since $\overleftrightarrow{GH}$ is parallel to $\overleftrightarrow{EF}$ by the definition of a parallelogram, and since $\overleftrightarrow{GH}$ is distinct from $\overleftrightarrow{EF}$ by the definition of a quadrilateral, we conclude that G and H lie on the same side of $\overleftrightarrow{EF}$. With the aid of Theorem 4-2, we find that one side of the line $\overleftrightarrow{EF}$ contains all the rest of the parallelogram. Similar remarks apply to any side of the parallelogram. Thus every parallelogram is a convex quadrilateral.



THEOREM 6-6. In any parallelogram, each side is congruent to the side opposite it.

Proof: Let the parallelogram be ABCD .
Hypotheses: $\overline{BA} \parallel \overline{DC}$ and $\overline{CB} \parallel \overline{AD}$.
Conclusion: $\overline{BA} \cong \overline{DC}$ and $\overline{CB} \cong \overline{AD}$.



Consider the two triangles ACB and CAD , which have the diagonal $\overline{AC}$ of the parallelogram as a common side.

Statements	Reasons
1. $\angle BAC \cong \angle DCA$.	1. Alternate interior angles, determined by the transversal $\overleftrightarrow{AC}$ of the parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are congruent.
2. $\overline{AC} \cong \overline{CA}$.	2. Why?
3. $\angle ACB \cong \angle CAD$.	3. Alternate interior angles, determined by the transversal $\overleftrightarrow{AC}$ of the parallel lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$, are congruent.
4. $\triangle ACB \cong \triangle CAD$.	4. Why?
5. $\overline{BA} \cong \overline{DC}$ and $\overline{CB} \cong \overline{AD}$.	5. Why?

The preceding theorem tells us that in a parallelogram, each pair of opposite sides are not only parallel but also congruent. The next result provides another method for establishing that a quadrilateral is a parallelogram. According to the following theorem, if just one pair of opposite sides of a quadrilateral are known to be both parallel and congruent, then the other pair of opposite sides must also be both parallel and congruent.

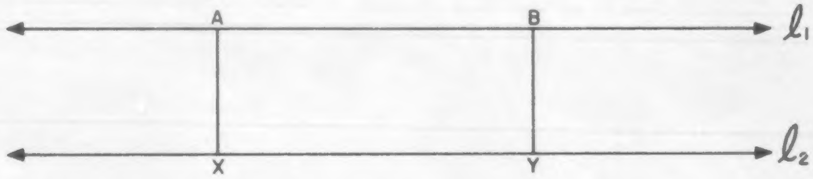
THEOREM 6-7. If two sides of a quadrilateral are parallel and congruent, then the quadrilateral is a parallelogram.

We leave the proof as a problem.

Our first formal introduction to "distance" was in Chapter 3. There Postulate 10 described the distance between a pair of points. Later we learned that the distance between two distinct points can also be called the length of the segment joining the points. We can now extend the notion of distance. We wish to examine the meaning of the distance between two parallel lines.

THEOREM 6-8. To every pair of distinct parallel lines there is a number which is the common length of all segments which have their respective endpoints on the given lines and are perpendicular to each of the given lines.

Proof: Let ℓ_1 and ℓ_2 be the two given parallel lines. Let A and B be points of ℓ_1 and let X and Y be points of ℓ_2 such that each of the distinct segments $\overline{AX}$ and $\overline{BY}$ is perpendicular to one of the given lines. By Corollary 6-4-3, each of the segments is also perpendicular to the other given line. By Theorem 6-1, $\overline{AX}$ and $\overline{BY}$ are parallel. Thus the quadrilateral ABYX is a parallelogram. By Theorem 6-6, AX = BY .



This theorem establishes the fact that there is a unique number which is the length of all perpendicular segments joining ℓ_1 and ℓ_2 , that is, segments which join a point of ℓ_1 and a point of ℓ_2 and are perpendicular to both ℓ_1 and ℓ_2 . This leads to the following definition.

DEFINITION. The distance between two distinct parallel lines is the length of a segment whose respective endpoints lie on the two parallel lines and which is perpendicular to both lines.

The distance between a line and itself is defined to be zero.

We sometimes express Theorem 6-8 by saying that two parallel lines are "everywhere equidistant."

The relation of parallelism is useful not only for lines and segments, but also for rays.

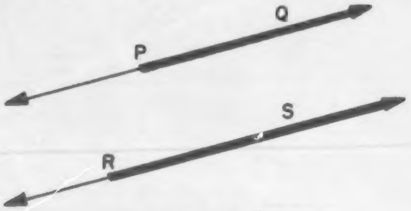


Figure (a)

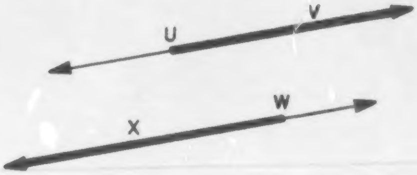


Figure (b)

In Figure (a), the two rays $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ lie on parallel lines, and we are tempted to say that the rays are parallel. In Figure (b), the two rays $\overrightarrow{UV}$ and $\overrightarrow{WX}$ lie on parallel lines. Again we may be tempted to call the rays parallel; but, on careful examination, there appears to be an important feature which distinguishes this pair of rays from the pair shown in Figure (a). In Figure (a), the two rays seem to extend into a single halfplane, namely a halfplane whose edge contains the respective endpoints P and R . Figure (c) shows the same rays as Figure (a) and also shows a halfplane with $\overleftrightarrow{PR}$ as

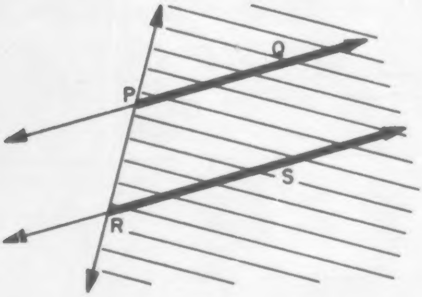


Figure (c)

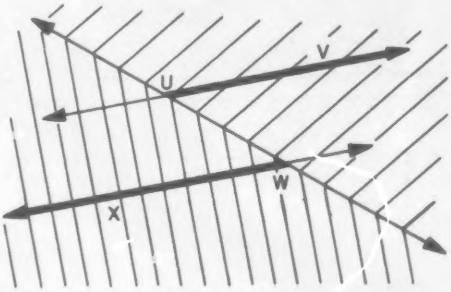
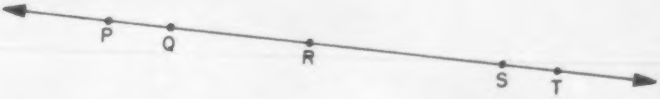


Figure (d)

edge. Figure (d) shows the same rays as Figure (b) and also shows the halfplanes which have $\overleftrightarrow{UW}$ as edge. We note that the two given rays $\overrightarrow{UV}$ and $\overrightarrow{WX}$ appear to extend into different halfplanes. The distinction suggested by the contrasting situations discussed above leads to the following definitions.

> DEFINITIONS. Two noncollinear rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said to be parallel if and only if the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel and furthermore B and D lie on the same side of the line $\overleftrightarrow{AC}$ determined by the respective endpoints of the rays. Two collinear rays are said to be parallel if and only if one of them is a subset of the other.
>
> Two noncollinear rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said to be antiparallel if and only if the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel and furthermore B and D lie on opposite sides of the line $\overleftrightarrow{AC}$ determined by the respective endpoints of the rays. Two collinear rays are said to be antiparallel if and only if neither is a subset of the other.

Let P, Q, R, S, T be five points, collinear in that order.



Then the collinear rays $\overrightarrow{QS}$ and $\overrightarrow{RT}$ are parallel, in accordance with the definition. The collinear rays $\overrightarrow{QS}$ and $\overrightarrow{TR}$ are a pair of antiparallel rays on the line. Another pair of antiparallel rays are $\overrightarrow{QP}$ and $\overrightarrow{ST}$. Also, $\overrightarrow{RP}$ and $\overrightarrow{RT}$ are antiparallel rays. The intersection of two antiparallel collinear rays may be a segment or may consist of a single point or may be the empty set.

Problem Set 6-7

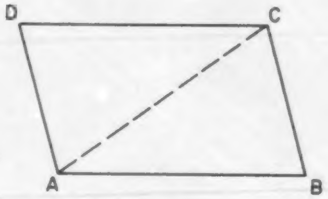
1. Supply the missing reasons for the proof of Theorem 6-6.

*2. Prove that each pair of opposite angles of a parallelogram are congruent.

3. Prove that the diagonals of a parallelogram bisect each other.

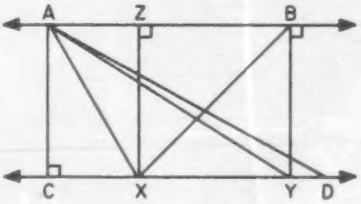
4. Prove Theorem 6-7.

   Hint: By hypothesis, two sides, say $\overline{AB}$ and $\overline{CD}$, of quadrilateral ABCD are both parallel and congruent. To prove that ABCD satisfies the definition of a parallelogram, try to show that $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$.



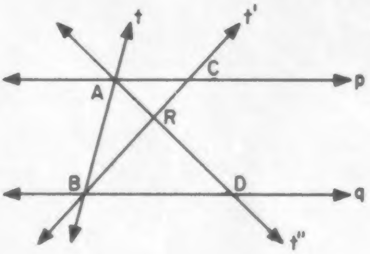
*5. Prove that quadrilateral ABCD is a parallelogram if $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{CB}$.

6. In the plane figure to the right, name the segments which picture the distance between $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ if $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.



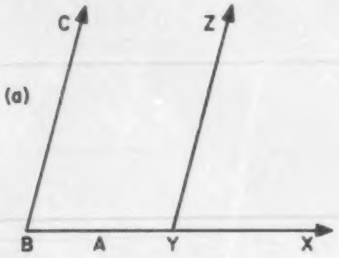
7. Given a transversal of two parallel lines. Prove that the angle bisectors of a pair of alternate interior angles are antiparallel rays.

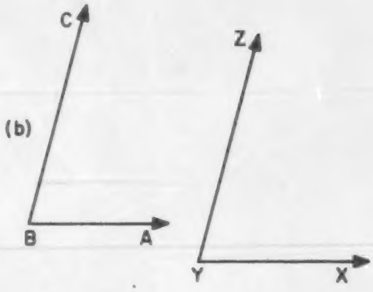
8. In the figure to the right, t, t' and t'' are transversals of p and q. t intersects p at A and q at B; t' intersects p at C and q at B; t'' intersects p at A and q at D. $\overrightarrow{AD}$ bisects $\angle CAB$ and $\overrightarrow{BC}$ bisects $\angle DBA$. $\overline{BC} \perp \overline{AD}$ at point R. Prove p is parallel to q.



*9. Let the coplanar angles $\angle ABC$ and $\angle XYZ$ have the property that the sides of one angle are respectively parallel (in the sense of "parallel rays") to the corresponding sides of the other angle. Prove that the angles are congruent if $\overrightarrow{BC}$ and $\overrightarrow{YZ}$ are noncollinear parallel rays and:

(a) $\overrightarrow{BA}$ and $\overrightarrow{YX}$ are collinear;
(b) $\overrightarrow{BA}$ and $\overrightarrow{YX}$ are not collinear.





*10. Let the coplanar angles $\angle ABC$ and $\angle XYZ$ have the property that the sides of one angle are respectively antiparallel to the corresponding sides of the other angle. Prove that the angles are congruent if $\overrightarrow{BC}$ and $\overrightarrow{YZ}$ are noncollinear antiparallel rays and:

(a) $\overrightarrow{BA}$ and $\overrightarrow{YX}$ are collinear;
(b) $\overrightarrow{BA}$ and $\overrightarrow{YX}$ are not collinear.

*11. Let the sides of the coplanar angles $\angle ABC$ and $\angle XYZ$ have the property that $\overrightarrow{BA}$ and $\overrightarrow{YX}$ are distinct parallel rays, and $\overrightarrow{BC}$ and $\overrightarrow{YZ}$ are antiparallel rays. Prove that the angles are supplementary if:

(a) $\overrightarrow{BA}$ and $\overrightarrow{YX}$ are collinear, and $\overrightarrow{BC}$ and $\overrightarrow{YZ}$ are noncollinear;
(b) $\overrightarrow{BA}$ and $\overrightarrow{YX}$ are noncollinear, and $\overrightarrow{BC}$ and $\overrightarrow{YZ}$ are collinear;
(c) $\overrightarrow{BA}$ and $\overrightarrow{YX}$ are noncollinear, and $\overrightarrow{BC}$ and $\overrightarrow{YZ}$ are noncollinear.

*12. The results of Problems 9, 10, and 11 may be summarized as follows:

If the sides of one angle are in one-to-one correspondence with the sides of another angle, then

(a) the angles are _____ if the corresponding sides are _____ .

(b) the angles are _____ if the corresponding sides are _____ .

(c) the angles are _____ if one pair of sides are _____ and the other pair of sides are _____ .

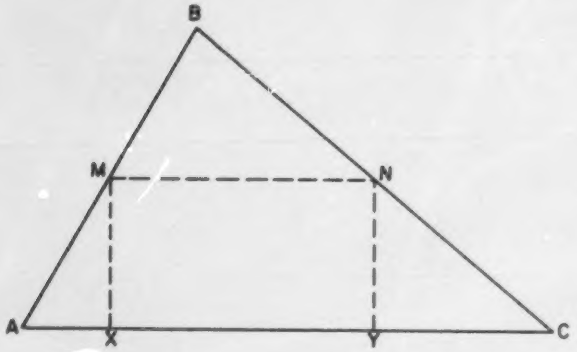
6-8. Sum of the Measures of the Angles of a Triangle.

In Problems 3 and 4 of Problem Set 1-4 you were asked to make inductions from experimental data. By measuring with a physical protractor each of the angles of a triangle or a quadrilateral in the real world, you were led to conclude, by inductive reasoning, that the sum of the measures of the angles is 180 for a triangle and is 360 for a quadrilateral. In this section we wish to establish these results by deductive reasoning.

Before we prove our main theorem, you should perform the following experiment.

Experiment

Make a large paper model of a triangle. Call this triangle ABC . Mark the midpoint M of $\overline{AB}$ and the midpoint N of $\overline{BC}$. Draw a line containing M perpendicular to $\overleftrightarrow{AC}$, and draw a line containing N perpendicular to $\overleftrightarrow{AC}$. Let these lines intersect $\overline{AC}$ at points X and Y , respectively. Draw $\overline{MN}$; your model should look like the following picture.



Fold the paper triangle along the segments $\overline{XM}$, $\overline{MN}$, $\overline{NY}$.

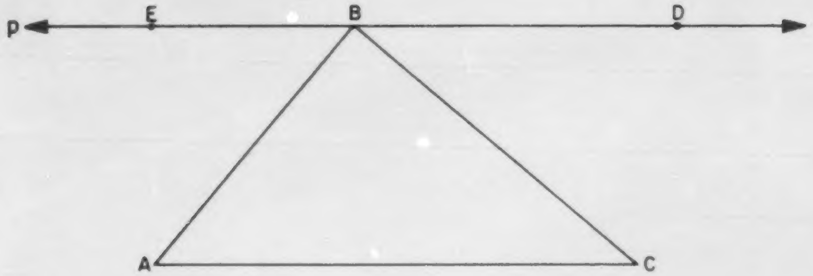
Do the vertices of the triangles seem to meet at a point?

Does this experiment help you to make a guess about the sum of the measures of the angles of a triangle? Explain why you made this guess.

The Parallel Postulate and the theorems which we proved about parallel lines provide us with sufficient information to deduce that the sum of the measures of the angles of a triangle is 180 .

THEOREM 6-9. The sum of the measures of the angles of a triangle is 180 .

Proof: Let the triangle have vertices A, B, C . There is a line, say p , which is parallel to $\overleftrightarrow{AC}$ and contains B . Hence p and $\overleftrightarrow{AB}$ have only the point B in common. Let D be any point of p on the same side of $\overleftrightarrow{AB}$ as C . Let E be any point of p on the opposite side of $\overleftrightarrow{AB}$ from C . Then $\overrightarrow{BE}$ and $\overrightarrow{BD}$ are opposite rays.



Since C and D are on the same side of $\overleftrightarrow{AB}$ and since C and A are on the same side of $\overleftrightarrow{BD}$, Postulate 16 enables us to conclude that $\overrightarrow{BC}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BD}$. By Theorem 4-9,

$$(1) \qquad m\angle DBC + m\angle CBA + m\angle ABE = 180 .$$

Since D and A are on opposite sides of $\overleftrightarrow{BC}$, $\angle DBC$ and $\angle BCA$ are a pair of alternate interior angles determined by the transversal $\overleftrightarrow{BC}$ of the two parallel lines. Hence,

$$(2) \qquad m\angle DBC = m\angle BCA .$$

Since E and C are on opposite sides of $\overleftrightarrow{BA}$, $\angle EBA$ and $\angle BAC$ are a pair of alternate interior angles determined by the transversal $\overleftrightarrow{BA}$ of the two parallel lines. Hence,

$$(3) \qquad m\angle EBA = m\angle BAC .$$

By the substitution property, we combine statements (1), (2), (3) to obtain

$$m\angle BCA + m\angle CBA + m\angle BAC = 180 ,$$

as we wished to prove.

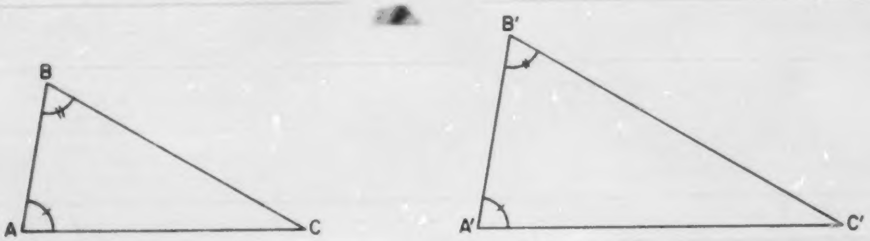
It is interesting to note that the sum of the measures of the angles of a triangle in non-Euclidean geometry is not 180 . In Lobachevskian geometry the sum is proved to be less than 180 , while in Riemannian geometry the sum is more than 180 .

The theorem pertaining to the sum of the measures of the angles of a triangle in Euclidean geometry has many useful consequences. Among them are the next three theorems, the proofs of which are left as problems.

THEOREM 6-10. The measure of an exterior angle of a triangle is equal to the sum of the measures of its non-adjacent interior angles.

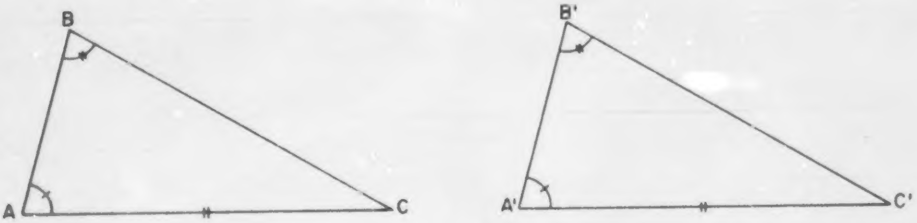
Note that Theorem 6-10 gives more specific information than Theorem 5-10 about the measure of an exterior angle of a triangle. Theorem 5-10 involves an inequality, while Theorem 6-10 involves an equality. The additional information has been made possible by our adoption of the Parallel Postulate.

THEOREM 6-11. Given a one-to-one correspondence between the vertices of two triangles, if two pairs of corresponding angles are congruent, then the third pair of corresponding angles are congruent.



The theorem says that if $\angle A \cong \angle A'$ and $\angle B \cong \angle B'$, then $\angle C \cong \angle C'$. As the figure suggests, the theorem applies to cases where the given correspondence is not a congruence as well as to cases where it is.

THEOREM 6-12. (The S.A.A. Theorem) Given a one-to-one correspondence between the vertices of two triangles, if two angles and a side opposite one of them in one triangle are congruent to the corresponding parts of the second triangle, the correspondence is a congruence.



Problem Set 6-8a

1. If the measures of two angles of a triangle are as follows, what is the measure of the third angle?

   (a) 37 and 58 .
   (b) 149 and 30 .
   (c) n and n .
   (d) r and s .
   (e) $45 + a$ and $45 - a$.
   (f) 90 and $\frac{1}{2}k$.

2. The measure of one of the base angles of an isosceles triangle is 72 . Find the measure of the vertex angle of the triangle.

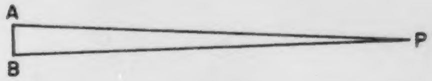
3. The measure of the vertex angle of an isosceles triangle is 72 . Find the measure of each base angle.

4. The measures of the angles of a triangle are in the ratio of 2 to 3 to 5 . Find the measure of each angle of the triangle.

   Hint: Let the measures of the angles be $2x$, $3x$, and $5x$.

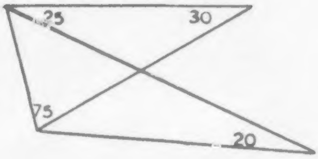
5. Find the measures of the three angles of a triangle if the measure of the second angle is three times the measure of the first, and the measure of the third angle is 12 less than twice the measure of the first.

6. To find the distance from a point A to a distant point P , a surveyor may measure a small distance AB and also measure $\angle A$ and $\angle B$. From this information he can compute the measure of $\angle P$, and by appropriate formulas then compute AP . If $m\angle A = 87.5$ and $m\angle B = 88.3$, compute $m\angle P$.



7. Why is the Parallel Postulate essential to the proof of Theorem 6-9?

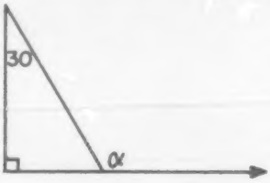
8. Copy the plane figure shown on the right, and fill in the measures of all of the angles.



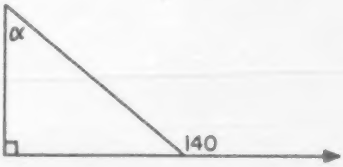
9. Prove Theorem 6-10.

10. Find the measure of $\angle\alpha$ in each of the following diagrams.

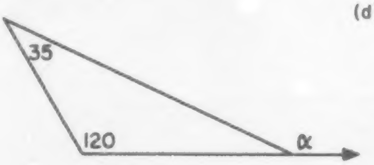
(a)



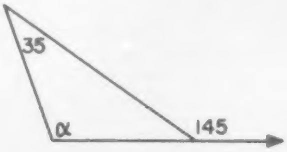
(b)



(c)



(d)



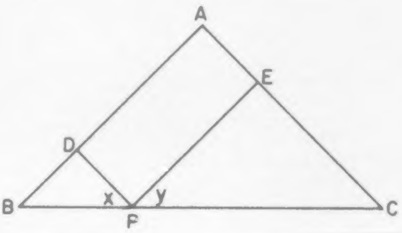
11. If triangle ABC is isosceles, find the measure of an exterior angle at the vertex if the measure of each of the base angles is 54 .

12. What is the sum of the measures of the exterior angles (one at each vertex) of an equilateral triangle?

13. Prove Theorem 6-11.

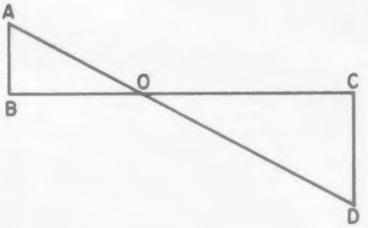
14. Given: $\triangle ABC$ is isosceles with base $\overline{BC}$. P is any point between B and C . $\overline{PD} \perp \overline{AB}$ and $\overline{PE} \perp \overline{AC}$.

Prove: $\angle x \cong \angle y$.



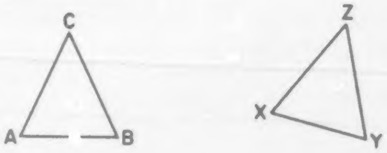
15. In the figure to the right $\overline{AD}$ and $\overline{BC}$ intersect at O . $\overline{AB} \perp \overline{BC}$; $\overline{CD} \perp \overline{BC}$.

    Prove: $\angle A \cong \angle D$.



16. Hypothesis: $\angle A \cong \angle X$ and $\angle B \cong \angle Y$. Which of the following statements (if either) can you correctly conclude? Explain why.

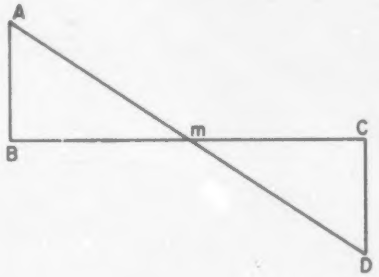
    (a) $\angle C \cong \angle Z$.
    (b) $\overline{AB} \cong \overline{XY}$.



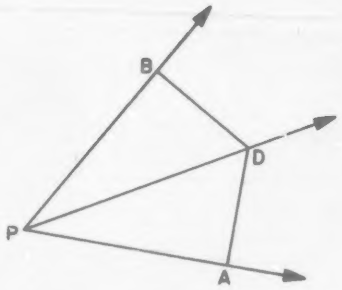
17. Prove Theorem 6-12.

18. In the figure to the right $\overline{AB} \perp \overline{BC}$; $\overline{DC} \perp \overline{BC}$; $\overline{AB} \cong \overline{DC}$.

    Prove: $\overline{AD}$ and $\overline{BC}$ bisect each other.

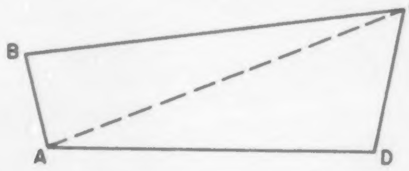


19. Let $\overrightarrow{PD}$ be between $\overrightarrow{PA}$ and $\overrightarrow{PB}$; let $\overline{DA} \perp \overrightarrow{PA}$ and $\overline{DB} \perp \overrightarrow{PB}$. If $\overrightarrow{PD}$ bisects $\angle BPA$, prove that $\overline{AD} \cong \overline{BD}$.



THEOREM 6-13. The sum of the measures of the angles of a convex quadrilateral is 360 .

Proof: Let the quadrilateral ABCD be a convex quadrilateral.



Using the triangle ABC , we find that

(1) $m\angle ABC + m\angle BCA + m\angle CAB = 180$.

Using the triangle ACD , we find that

(2) $m\angle ACD + m\angle CDA + m\angle DAC = 180$.

From (1) and (2) by addition,

(3) $m\angle ABC + m\angle BCA + m\angle ACD + m\angle CDA + m\angle DAC + m\angle CAB = 360$.

Since ABCD is a convex quadrilateral, $\overrightarrow{CA}$ is between $\overrightarrow{CB}$ and $\overrightarrow{CD}$, and hence

(4) $m\angle BCA + m\angle ACD = m\angle BCD$.

Similarly,

(5) $m\angle DAC + m\angle CAB = m\angle DAB$.

By substitution from (4) and (5) into (3), we conclude that

$$m\angle ABC + m\angle BCD + m\angle CDA + m\angle DAB = 360 ,$$

as asserted.

Problem Set 6-8b

1. If the sum of the measures of three angles of a convex quadrilateral is 300 , find the measure of the fourth angle.
2. If the sum of the measures of three angles of a convex quadrilateral is 270 , find the measure of the fourth angle. Is the quadrilateral equiangular? Explain your answer.

3. ABCD is a parallelogram with $\overline{AB} \perp \overline{BC}$. Find the measure of each angle of the parallelogram.

4. ABCD is a convex quadrilateral. $\angle A \cong \angle C$ and $\angle B \cong \angle D$. If $m \angle A = 72$, find the measures of the other three angles of the quadrilateral.

*5. Prove the following: If each angle of a convex quadrilateral is congruent to the angle opposite it, then the quadrilateral is a parallelogram.

Hint: You will need to use Theorem 6-13 and the definition of a parallelogram in your proof.

6-9. Right Triangles.

A proof of the following theorem was asked for in Problems 12 and 13 of Problem Set 5-12a.

THEOREM 6-14. If one of the angles of a triangle is a right angle or an obtuse angle, then each of the other angles is an acute angle.

A highly important type of triangle is the triangle such that the measure of one of its angles is 90. On the basis of the preceding theorem, a triangle can have at most one right angle. Consequently we are ready for the following definitions, which describe a right triangle and its sides. Read again, at this time, the discussion concerning a right triangle in Section 1-6. Note that now we have acquired an appreciation of the meaning of the various words used in the definition.

DEFINITION. A triangle one of whose angles is a right angle is called a right triangle.

The side of a right triangle opposite the right angle is called the hypotenuse of the triangle.

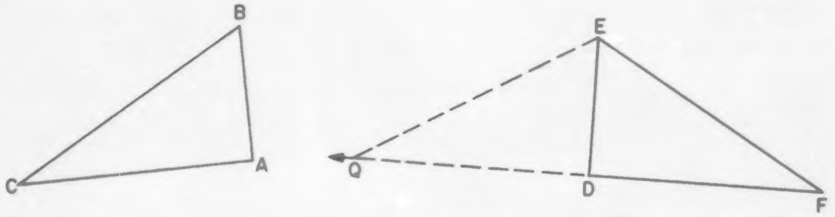
Each of the two sides of a right triangle which are contained in the right angle is called a leg of the triangle.

The proof of the next theorem is left as a problem.

THEOREM 6-15. The acute angles of a right triangle are complementary.

THEOREM 6-16. (The Hypotenuse-Leg Theorem) Let a one-to-one correspondence between the vertices of two right triangles have the property that the vertices of the respective right angles correspond. If the hypotenuse and one leg of one triangle are congruent to the corresponding parts of the other triangle, then the correspondence is a congruence.

Proof: Let one right triangle be $\triangle ABC$ with $m\angle A = 90$, let the other right triangle be $\triangle DEF$ with $m\angle D = 90$, let $ABC \leftrightarrow DEF$ be the given correspondence having the properties that $\overline{BC} \cong \overline{EF}$ and $\overline{AB} \cong \overline{DE}$. We must prove that the correspondence is a congruence.



Statements	Reasons
1. On the ray opposite to $\overrightarrow{DF}$, there is a point Q such that $\overline{CA} \cong \overline{QD}$.	1. Point-Plotting Theorem.
2. $m\angle QDE = 90$.	2. Why?
3. $\angle CAB \cong \angle QDE$.	3. Why?
4. $\overline{AB} \cong \overline{DE}$.	4. By hypothesis.
5. $\triangle ABC \cong \triangle DEQ$.	5. S.A.S.
6. $\overline{BC} \cong \overline{EQ}$.	6. Why?
7. $\overline{BC} \cong \overline{EF}$.	7. By hypothesis.
8. $\overline{EQ} \cong \overline{EF}$.	8. Why?
9. $\angle EQD \cong \angle EFD$.	9. The base angles of the isosceles triangle EQF are congruent.
10. $\angle QDE \cong \angle FDE$.	10. Why?
11. $\triangle DEQ \cong \triangle DEF$.	11. S.A.A.
12. $\triangle ABC \cong \triangle DEF$.	12. Transitive property of triangle congruence.

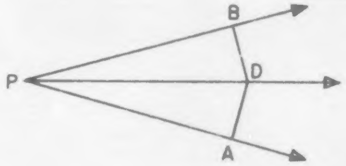
Problem Set 6-9

1. Find the measure of each of the acute angles of an isosceles right triangle.

2. The measure of one of the acute angles of a right triangle is 72. Find the measure of the other acute angle.

3. Prove Theorem 6-15.

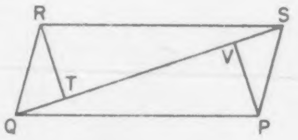
4. Given $\angle APB$ with $\overrightarrow{PD}$ between $\overrightarrow{PA}$ and $\overrightarrow{PB}$. Let $\overline{DA} \perp \overrightarrow{PA}$ and $\overline{DB} \perp \overrightarrow{PB}$. If $\overline{BD} \cong \overline{AD}$, prove $\overrightarrow{PD}$ is the midray of $\angle BPA$.



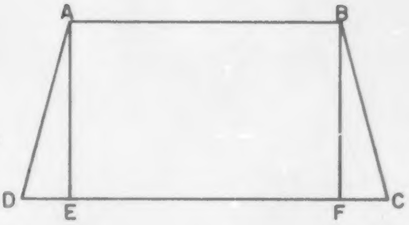
5. In quadrilateral $PQRS$; $\overline{RT} \perp \overline{QS}$; $\overline{PV} \perp \overline{QS}$; $\overline{RQ} \cong \overline{PS}$; $\overline{RT} \cong \overline{PV}$.

   Prove: $\overline{QT} \cong \overline{SV}$

   $\triangle PQV \cong \triangle RST$.



6. In quadrilateral ABCD, let $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$ and let E and F be points between D and C such that $\overline{DC}$ is perpendicular to both $\overline{AE}$ and $\overline{BF}$.



   (a) If $AD = BC$, prove that $\angle D \cong \angle C$ and $\angle DAB \cong \angle CBA$.

   (b) If $\angle D \cong \angle C$, prove that $AD = BC$.

   Hint: Why does $AE = BF$?

7. If $\overline{AD}$ and $\overline{BE}$ are altitudes of $\triangle ABC$ and $AD = BE$, then $\triangle ABC$ is isosceles.

   Consider three cases:

   (a) If $\overrightarrow{AD} = \overrightarrow{AC}$ and $\overrightarrow{BE} = \overrightarrow{BC}$.

   (b) If $\overrightarrow{AD}$ lies between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and $\overrightarrow{BE}$ lies between $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

   (c) If $\overrightarrow{AD}$ does not lie between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and $\overrightarrow{BE}$ does not lie between $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

6-10. Inequalities in the Same Order.

In this section we are concerned with angles having unequal measures and with segments having unequal measures. In comparing two segments whose lengths are unequal, we naturally say that the segment with the greater measure is longer than the other; similarly, a segment whose length is less than the measure of another segment is said to be shorter than the other segment. The words "longest" or "shortest" are of course applicable when the measure of more than two segments are compared.

Let $S = \{p,q\}$ and $T = \{u,v\}$ be two sets of real numbers. Consider the one-to-one correspondence between these two sets given by: $p \longleftrightarrow u$, $q \longleftrightarrow v$. Suppose the numbers of S satisfy the condition $p > q$ and that the corresponding numbers in T satisfy the condition $u > v$. Then we may say that the numbers in S are "unequal in the same order" as the corresponding numbers in T.

For example, if a correspondence between $\{2,1\}$ and $\{10,0\}$ is given by

$$\begin{array}{c} 2 \longleftrightarrow 10 \\ 1 \longleftrightarrow 0 \ ; \end{array}$$

then the numbers $2,1$ are unequal in the same order as the corresponding numbers $10,0$ because $2 > 1$ and $10 > 0$.

The remaining correspondence between $\{2,1\}$ and $\{10,0\}$ is given by

$$\begin{array}{c} 2 \longleftrightarrow 0 \\ 1 \longleftrightarrow 10 \ . \end{array}$$

In this correspondence, the numbers in $\{2,1\}$ and $\{10,0\}$ are not unequal in the same order because $2 > 1$ and $0 < 10$.

Consider, as another example, the correspondences between $\{6,12\}$ and $\{9,4\}$ which are

$$\begin{array}{c} 6 \longleftrightarrow 4 \\ 12 \longleftrightarrow 9 \end{array} \quad \text{and} \quad \begin{array}{c} 6 \longleftrightarrow 9 \\ 12 \longleftrightarrow 4 \ . \end{array}$$

In the first correspondence the numbers in $\{6,12\}$ and $\{9,4\}$ are unequal in the same order because $6 < 12$ and $4 < 9$. In the second correspondence the numbers are not unequal in the same order because $6 < 12$ and $9 > 4$.

This notion of inequality is formulated in the following definition.

> DEFINITION. Let $\{x,y\}$ be a set of real numbers such that $x > y$, let $\{x',y'\}$ be a set of real numbers, let C be a one-to-one correspondence between $\{x,y\}$ and $\{x',y'\}$ such that $x \longleftrightarrow x'$ and $y \longleftrightarrow y'$; the numbers of $\{x,y\}$ are said to be unequal in the same order as the corresponding numbers of $\{x',y'\}$ if and only if $x' > y'$.

Problem Set 6-10

1. Given $\{10,3\}$ and $\{8,4\}$. Identify the following as true or false:

   The numbers 10,3 are unequal in the same order as the corresponding numbers 8,4 if the correspondence is given by (a) $10 \longleftrightarrow 8$, $3 \longleftrightarrow 4$ (b) $10 \longleftrightarrow 4$, $3 \longleftrightarrow 8$

2. Given $\{x,y\}$ and $\{u,v\}$ and $x > y$. For each of the following is the statement "u and v are unequal in the same order as x and y" true or false if the correspondence is given by $x \longleftrightarrow u$ and $y \longleftrightarrow v$?

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
u	3	5	8	12	17	20	25
v	2	6	4	11	20	17	42

3. Answer Problem 2 if the correspondence is given by $x \longleftrightarrow v$ and $y \longleftrightarrow u$.

4. We sometimes wish to speak of a certain set of numbers as being unequal in the same order as the corresponding numbers of another set, in a situation where each set has more than two members. A natural extension of the definition for sets with two elements leads to the following:

   Let $\{x,y,z\}$ be a set of real numbers such that $x > y > z$, let C be a one-to-one correspondence between the set $\{x,y,z\}$ and the set $\{x',y',z'\}$ of real numbers such that $x \longleftrightarrow x'$, $y \longleftrightarrow y'$, $z \longleftrightarrow z'$; the numbers of the set $\{x,y,z\}$ are said to be unequal in the same order as the corresponding numbers of the set $\{x',y',z'\}$ if and only if $x' > y' > z'$.

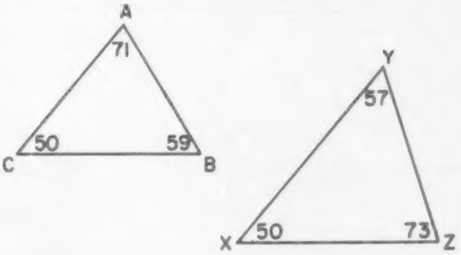
   List all the correspondences between $\{x,y,z\}$ and $\{x',y',z'\}$.

5. Which, if any, of the following correspondences has the property that the numbers of the set $\{6,5,4\}$ are unequal in the same order as the corresponding numbers of the set $\{45,60,75\}$?

(a) $6 \leftrightarrow 45$, $5 \leftrightarrow 60$, $4 \leftrightarrow 75$;
(b) $6 \leftrightarrow 60$, $5 \leftrightarrow 75$, $4 \leftrightarrow 45$;
(c) $6 \leftrightarrow 75$, $5 \leftrightarrow 60$, $4 \leftrightarrow 45$.

6. Which of the following correspondences has the property that the measures of the angles of one triangle are unequal in the same order as the measures of the corresponding angles in the second triangle?

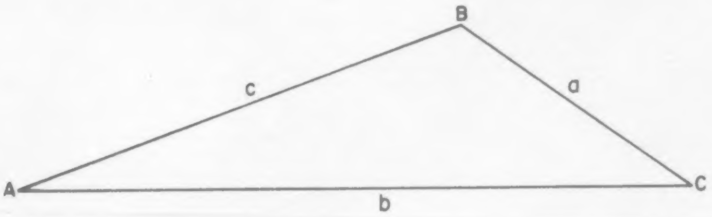
(a) $ABC \leftrightarrow XYZ$.
(b) $ABC \leftrightarrow XZY$.
(c) $ABC \leftrightarrow YXZ$.
(d) $ABC \leftrightarrow YZX$.
(e) $ABC \leftrightarrow ZXY$.
(f) $ABC \leftrightarrow ZYX$.



6-11. Inequalities Involving Triangles.

Measure the sides and the angles of the triangle below. Record the measurements in a table as shown.

Angle	A	B	C
Measure of angle			
Length of side opposite angle			



Observe the order relation among the measures of the angles of the triangle, and complete the following statement:

$$m\angle \_\_\_\_ > m\angle \_\_\_\_ > m\angle \_\_\_\_ .$$

Observe the order relation among the measures of the sides of the triangle, and complete the following statement:

$$\_\_\_\_ > \_\_\_\_ > \_\_\_\_ .$$

How does the order relation among the measures of the angles of a triangle compare with the order relation among the measures of the respectively opposite sides of the triangle?

Using the phrase "unequal in the same order as", write a sentence which compares the measures of $\angle A$ and $\angle B$ with the lengths of the sides opposite these angles. Do the same for $\angle C$ and $\angle A$. Using the phrase "unequal in the same order as", write a sentence which compares the numbers b, c with the measures of the angles opposite the sides whose lengths are b and c. Do the same for the numbers c and a.

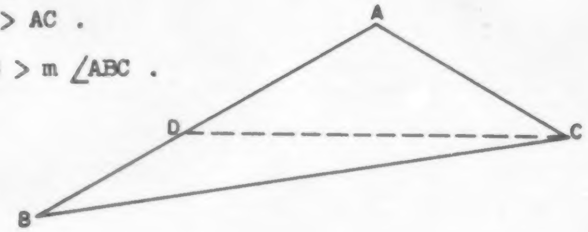
The comparisons suggested by the preceding experiment are formulated in the following two theorems.

THEOREM 6-17. If the lengths of two sides of a triangle are unequal, then the measures of the angles opposite these sides are unequal in the same order.

381

Proof: Given $\triangle ABC$,
$AB > AC$.

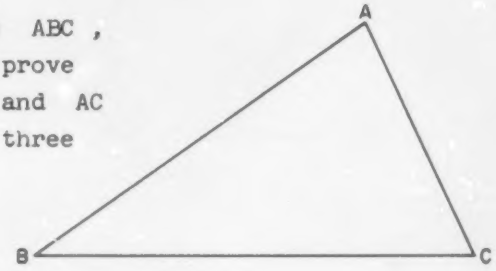
To Prove: $m\angle ACB > m\angle ABC$.



	Statements		Reasons
1.	There is a point D on $\overrightarrow{AB}$ such that $AD = AC$.	1.	The Point-Plotting Theorem.
2.	D is between A and B .	2.	Since $AC < AB$ by hypothesis, $AD < AB$.
3.	D is in the interior of $\angle ACB$.	3.	The Interior of an Angle Postulate.
4.	$m\angle ACB = m\angle ACD + m\angle DCB$.	4.	The Betweenness-Angles Theorem.
5.	$m\angle ACB > m\angle ACD$.	5.	The sum of two positive numbers is greater than either of them.
6.	$m\angle ACD = m\angle ADC$.	6.	If two sides of a triangle are congruent, the angles opposite are congruent.
7.	$m\angle ADC > m\angle ABC$.	7.	$\angle ADC$ is an exterior angle of $\triangle DCB$ at D .
8.	$m\angle ACB > m\angle ABC$.	8.	Transitive property of order (applied to Steps 5, 6, 7).

We now state and prove the converse of the preceding theorem.

THEOREM 6-18. If the measures of two angles of a triangle are unequal, then the lengths of the sides opposite these angles are unequal in the same order.



Proof: In the triangle ABC, let $m\angle C > m\angle B$. We must prove that $AB > AC$. Since AB and AC are numbers, there are only three possibilities:

(1) $AB = AC$,
(2) $AB < AC$, or
(3) $AB > AC$.

The method of the proof is to show that the first two of these "possibilities" are in fact impossible. The only remaining possibility then is (3), and this proves the theorem.

(1) If $AB = AC$, then by Theorem 5-6 it follows that $\angle C \cong \angle B$; and this contradicts the hypothesis. Therefore, it is impossible that $AB = AC$.

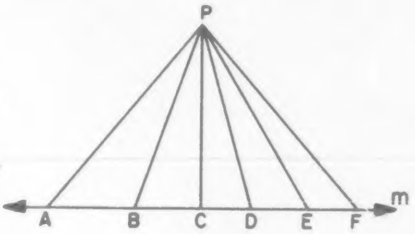
(2) If $AB < AC$, then by Theorem 6-17 it follows that $m\angle C < m\angle B$; and this contradicts the hypothesis. Therefore, it is impossible that $AB < AC$.

The only remaining possibility is that $AB > AC$.

The proof of the following corollary will be left as a problem.

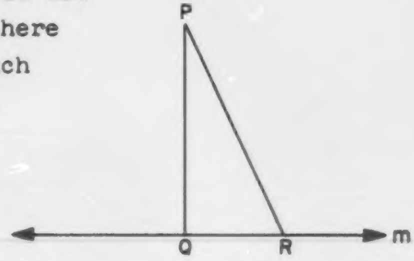
Corollary 6-18-1. The hypotenuse of a right triangle is the longest side of the triangle.

Let m be a line and P a point not on m. There are many segments joining P to m, in fact one for every point on m. One of these segments is perpendicular to m. The next theorem states a property which our experience leads us to expect, namely that the segment perpendicular to m is shorter than any other segment joining P to m.



THEOREM 6-19. The shortest segment joining a point to a line not containing the point is the segment perpendicular to the line.

Proof: Let m and P be the line and the point, respectively. By hypothesis, P is not on m. Hence, by Theorem 5-11, there is exactly one point Q on m such that $\overline{PQ}$ is perpendicular to m. Let R be any other point on m. The triangle PQR is a right triangle with right angle at Q. Thus $\overline{PQ}$ is shorter than the hypotenuse $\overline{PR}$ of the triangle. In other words, $\overline{PQ}$ is shorter than any other segment joining P to m.



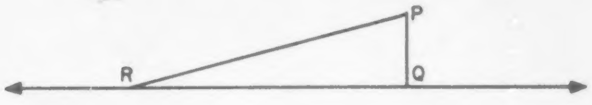
The preceding theorem suggests the notion of "how far apart" a line and a point are.

DEFINITION. The distance between a line and a point not on the line is the length of the perpendicular segment joining the point to the line.

The distance between a line and a point on the line is defined to be zero

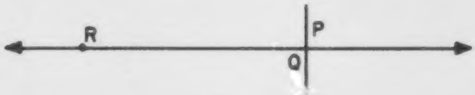
Corollary 6-19-1. If a line perpendicular to $\overleftrightarrow{QR}$ at the point Q contains a point P, then $PQ < PR$.

Proof: In case P is not on $\overleftrightarrow{QR}$,



the assertion is a restatement of Theorem 6-19.

In case P is on $\overleftrightarrow{QR}$,



then $P = Q$; hence $PQ = 0$, while $PR = QR > 0$, and consequently $PQ < PR$.

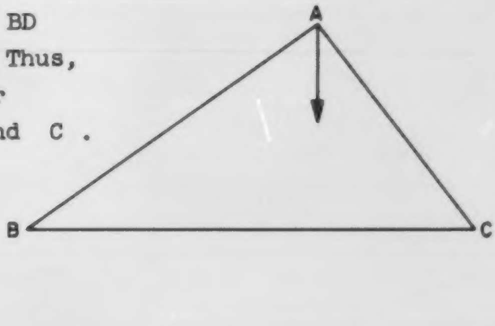
THEOREM 6-20. If the length of one side of a triangle is equal to or greater than the length of each of the other sides, then the perpendicular segment joining the opposite vertex to this side intersects this side in an interior point of the side.

Proof: Let the vertices of the triangle, A, B, C be named so that

(1) $$BC \geq BA \text{ and } BC \geq CA .$$

Let the perpendicular from A to $\overleftrightarrow{BC}$ intersect $\overleftrightarrow{BC}$ at D . We must prove that D is between B and C .

Now the line perpendicular to $\overleftrightarrow{DA}$ at D is $\overleftrightarrow{BC}$ and hence contains both B and C . We apply the preceding corollary twice. One application yields $BD < BA$. The other application yields $CD < CA$. These results, together with (1), tell us that each of the numbers BD and CD is less than BC . Thus, since B, C, D are collinear points, D is between B and C .



Experiment

Use a millimeter scale to measure the sides of each of the triangles below. Record your results in a data chart.

Figure	p	q	r	q + r
a				
b				
c				

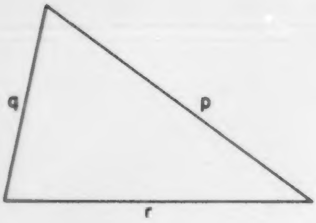
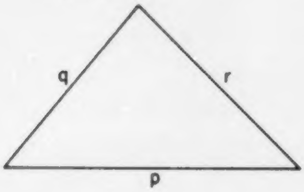


FIGURE a

FIGURE b



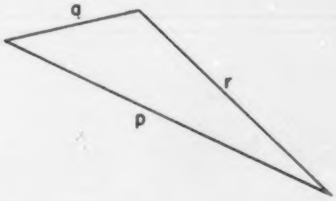


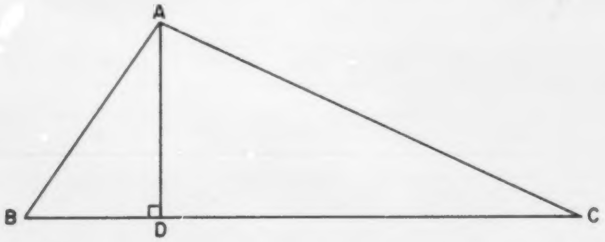
FIGURE c

How does the sum of q and r compare with p ?

Make a statement which expresses the relation of the sum of the lengths of any two sides of a triangle and the length of the third side of the triangle.

THEOREM 6-21. (The Triangle Inequality) The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

Proof: Given any triangle, there is one side which is at least as long as each of the other two sides. Why? Suppose the triangle is labeled ABC so that $BC \geq AB$ and $BC \geq AC$. Let the perpendicular from A to $\overleftrightarrow{BC}$ intersect $\overleftrightarrow{BC}$ in D . Then, by Theorem 6-20, D is an interior point of $\overline{BC}$.



Then

$AB > BD$. Why?
$AC > DC$. Why?

By adding we get

$AB + AC > BC$.

On the other hand, since $CB \geq AB$, it follows that $AC + CB > AB$. Similarly, $AB + BC > AC$.

Problem Set 6-11

*1. Prove Corollary 6-18-1.

2. Given $\triangle ABC$ with $AB = 10$, $BC = 12$, and $AC = 8$. Name the angles in order of size beginning with the angle with the least measure.

3. Name the longest side of $\triangle ABC$ if:

   (a) $m\angle A = 60$, $m\angle B = 90$.
   (b) $m\angle A = 110$, $m\angle B = 30$.
   (c) $m\angle A = 60$, $m\angle B = 50$.

4. Can a triangle exist with the following numbers as lengths of the sides?

   (a) 3 , 2 , 7 .
   (b) 3 , 4 , 7 .
   (c) 4 , 5 , 7 .

5. If the lengths of two sides of a triangle are 8 and 12 , the length of the third side of the triangle must be greater than _____ and less than _____ .

6. If the lengths of two sides of a triangle are 5 and 8 , the third side must have a length greater than _____ and less than _____ .

7. Choose the correct answer: The absolute value of the difference between the lengths of any two sides of a triangle is (greater than, equal to, less than) the length of the third side.

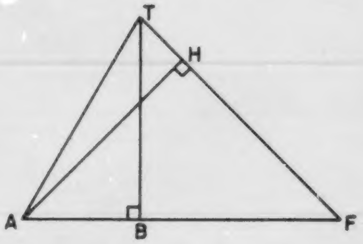
8. Suppose that you wish to draw a triangle with j as the length of one side and k as the length of a second side. It is known that $j > k$. What are the restrictions on the length, x, of the third side? Explain your answer.

9. In the figure to the right:

$AH <$ _____ and $AH <$ _____ .

$BT <$ _____ and $BT <$ _____ .

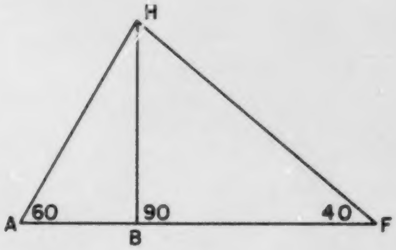
State the theorem involved.



10. Given the angle measures as shown in the figure, insert HA , HF , HB below in correct order.

_____ < _____ < _____ .

State theorems to support your conclusion.



11. In the figure to the right, $\overline{AB} \perp \overline{CD}$, $BD > BC$, $BE = BC$. Use $=$, $>$, or $<$ to complete correctly each of the following statements. Give a reason for each of your answers.

(a) AD _____ AB .

(b) $m\angle s$ _____ $m\angle D$.

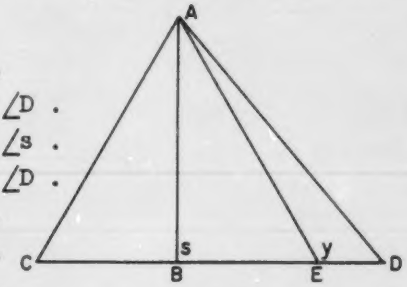
(c) $m\angle y$ _____ $m\angle s$.

(d) $m\angle y$ _____ $m\angle D$.

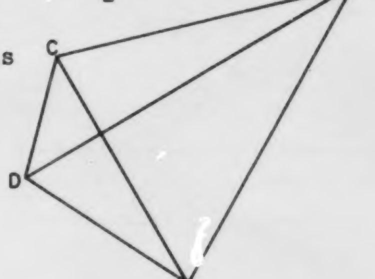
(e) AD _____ AE .

(f) AE _____ AC .

(g) AD _____ AC .



12. Prove that the sum of the lengths of the diagonals of a convex quadrilateral is less than the sum of the lengths of its four sides.



6-12. Summary.

Our geometry has been richly enhanced in this chapter by the adoption of the Parallel Postulate. Before the introduction of the Postulate, we proved several theorems whose conclusions are, "The given lines are parallel." The Postulate enabled us to prove the converses of the theorems. These results link the parallelism of two distinct coplanar lines with the properties of certain angles formed by the lines and a transversal of them.

The relation of parallelism for lines in a given plane is reflexive, symmetric, and transitive. The same properties apply to parallelism for segments in a given plane, or to parallelism for rays in a given plane.

A particularly important type of quadrilateral is the parallelogram. We have studied several properties of a parallelogram. We have found several different hypotheses concerning a quadrilateral, each of which yields the conclusion that the quadrilateral is a parallelogram. (You will be asked to make a list of these in a review problem.)

A striking consequence of our adoption of the Parallel Postulate is the theorem that, for every triangle, the sum of the measures of the angles is 180 . As we have already remarked, this "well-known fact" is not true in some other geometries.

Another useful by-product of the Parallel Postulate is concerned with distance. As an extension of our notion of the distance between two points, we have described the distance between two parallel lines. We also introduced (without using the Parallel Postulate) the distance between a line and a point.

A one-to-one correspondence between the sides and the angles of a triangle matches a side and an angle which are opposite each other. Under this type of correspondence, a longest side and an angle with greatest measure match; a shortest side and an angle with least measure match; two sides with equal lengths and two angles with equal measures match.

We may summarize by saying that the lengths of the three sides and the measures of the three angles are unequal (or equal) in the same order.

Finally the Triangle Inequality Theorem combines with our results in Chapter 3 to tell us that the distance between two distinct points A and B is less than the sum of the distances from A to Q and from Q to B for any point Q not on $\overleftrightarrow{AB}$. Thus the "straight path" from A to B is shorter than any other "path" along segments from A to B . This permits us to complete the discussion of a characterization mentioned in Chapter 3 (and a review problem asks you for the proof): namely, for any three distinct points A, B, C in space, $AB + BC = AC$ if and only if B is between A and C .

Our theory of similarity in the next chapter depends upon the relationship among the lengths of certain segments determined by a pair of parallel lines and two of their transversals.

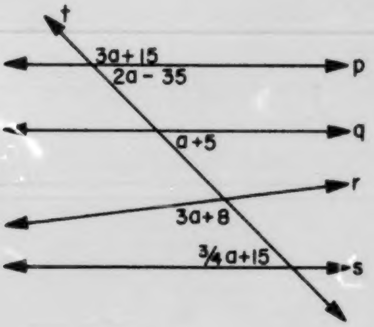
VOCABULARY LIST

parallel lines
parallel segments
parallel rays
antiparallel rays
transversal
alternate interior angles
consecutive interior angles
corresponding angles
parallelogram
right triangle
hypotenuse
leg
indirect method of proof
contrapositive

Review Problems

1. Write the contrapositive of Theorem 6-4 and each of the corollaries of Theorem 6-4. Can we accept these statements as true at this time? Why?

2. In the figure to the right, p, q, r, s are coplanar lines cut by transversal t. The measures of five of the angles are as designated in the picture. Which of the distinct lines are parallel?



3. Write true or false for each of the following:

   (a) Through a point not on a line there can be two lines parallel to the given line.

   Refer to Figure 1 for Parts (b) through (e).

   (b) If $m\angle e + m\angle g = 180$, then $\ell \parallel \ell'$.

   (c) If $\ell \parallel \ell'$, then $m\angle c = m\angle h$.

   (d) If $m\angle b = m\angle h$, then $\ell \parallel \ell'$.

   (e) If $m\angle f \neq m\angle g$, then ℓ is not parallel to ℓ'.

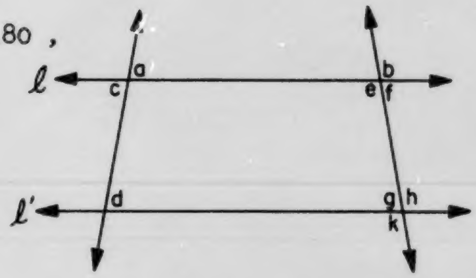


Figure 1

Refer to Figure 2 for Parts (f) and (g) and consider $t \parallel t'$.

(f) $m\angle z < m\angle x$.

(g) $m\angle z > m\angle r$.

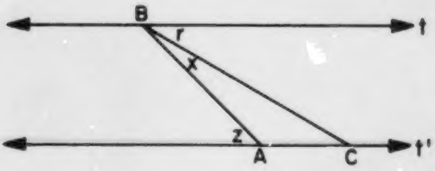


Figure 2

In Figure 3, $\angle R$ is a right angle of $\triangle RST$ and $RS > RT$.

(h) $m\angle T > m\angle S$.

(i) $m\angle T + m\angle S = 90$.

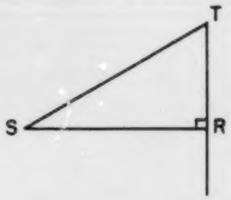


Figure 3

In Figure 4, $\overrightarrow{XW}$ is a midray of $\angle YXZ$ and $\overrightarrow{XW}$ intersects $\overline{YZ}$ at W .

(j) $\overline{XY} \cong \overline{XZ}$.

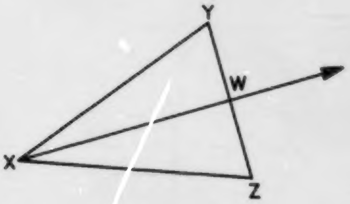
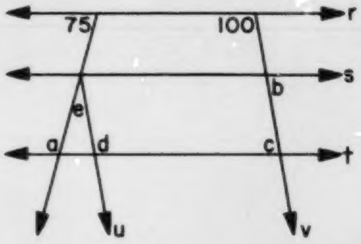
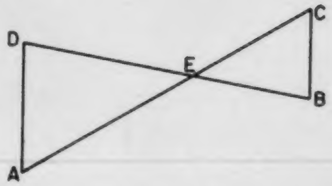


Figure 4

4. In the figure $r \parallel s \parallel t$ and transversals u and v are parallel. Note the given measures and find the measures of $\angle a$, $\angle b$, $\angle c$, $\angle d$, $\angle e$.



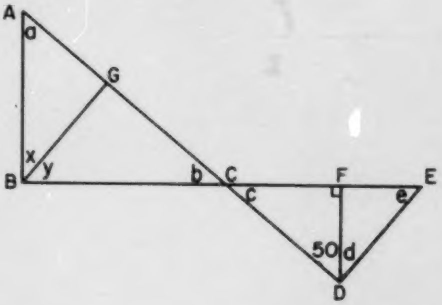
5. In this figure $\overline{AC}$ intersects $\overline{BD}$ at E so that $\angle C \cong \angle A$. State two reasons why $\angle B \cong \angle$ _____ .



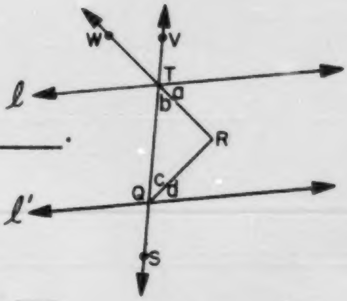
6. By hypothesis $\overline{AD}$ intersects $\overline{BE}$ in C . B, C, F, E are collinear in that order. $\overline{AB} \perp \overline{BE}$, $\overline{DF} \perp \overline{BE}$, $\overline{DE} \perp \overline{AD}$, $\overline{BG} \perp \overline{AD}$.

(a) Give the measures of the seven acute angles labeled in the drawing with lower case letters.

(b) Name the two pairs of noncollinear parallel segments.



7. In the plane figure at the right, $\ell \parallel \ell'$, m $\angle a$ = m $\angle b$, m $\angle c$ = m $\angle d$. Fill each blank with a single letter or a number.



(a) m $\angle a$ + m $\angle b$ + m $\angle c$ + m $\angle d$ = _____.

(b) m $\angle b$ + m $\angle c$ = _____ .

(c) m $\angle R$ = _____ .

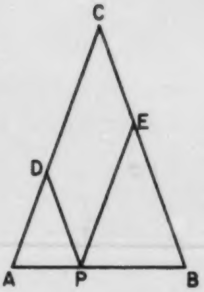
(d) If m $\angle c$ = 40 , m $\angle b$ = _____

(e) m $\angle RQS$ = m $\angle R$ + m $\angle$ _____ .

8. In the figure to the right, ABC is an isosceles triangle with base $\overline{AB}$; P is between A and B ; $\overline{PE} \parallel \overline{AC}$ and $\overline{PD} \parallel \overline{BC}$.

Prove that

PE + EC + CD + DP = AC + BC .

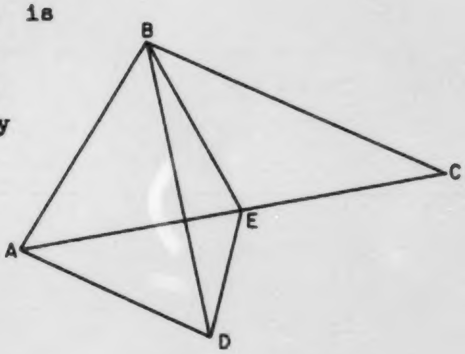


9. Given a transversal of two parallel lines. Prove that the angle bisectors of a pair of consecutive interior angles are perpendicular rays.

10. Given a transversal of two parallel lines. Prove that the bisectors of a pair of corresponding angles are parallel rays.

11. Given: $\triangle ABC$, point D is not on $\overleftrightarrow{AC}$, BD = BC, point E is between A and C, $\overrightarrow{BE}$ is midray of $\angle DBC$.

Prove: AC > AD.



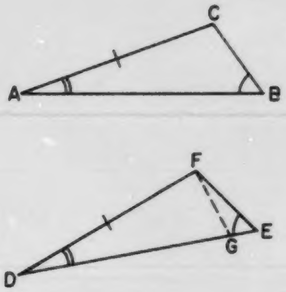
12. (a) Given points A, B, and C , not necessarily distinct.

Prove that $AB + BC \geq AC$.

(b) If A and C are two distinct points, prove that AB + BC = AC if and only if the point B is in $\overline{AC}$.

13. Use the indirect method to prove Theorem 6-12.

Hint: Let $ABC \leftrightarrow DEF$ be a correspondence such that $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\overline{AC} \cong \overline{DF}$. There is a point G on $\overrightarrow{DE}$ such that $\overline{DG} \cong \overline{AB}$. Why? Show that the correspondence $ABC \leftrightarrow DGF$ is a congruence. Then show that $\angle DGF \cong \angle DEF$. Why must E and G be the same point? Why is the correspondence $ABC \leftrightarrow DEF$ a congruence?



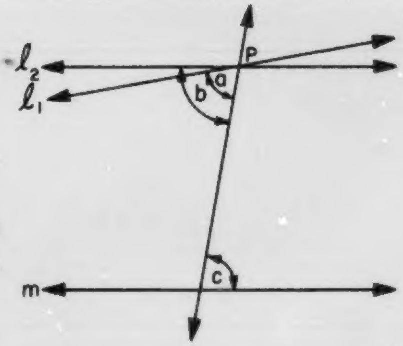
14. Consider the following two statements:

(I) There is at most one line parallel to a given line and containing a given point not on the given line.

(II) If two lines are parallel, then any two alternate interior angles determined by a transversal of the lines are congruent.

Our procedure has been to choose Statement I as a postulate (Postulate 22) and then, using that postulate and others, to prove Statement II as a theorem (Theorem 6-4). Instead of doing this, suppose you accept Statement II as a "postulate." On the basis of this and work before Section 6-5, prove Statement I.

Hint: Let m be a line and P a point not on m. You must prove there is at most one line containing P and parallel to m. Assume that there are two such lines, say ℓ_1 and ℓ_2. Try to reach a contradiction of Statement II.



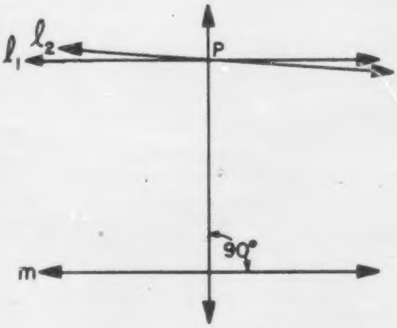
15. Consider the following two statements:

(I) There is at most one line parallel to a given line and containing a given point not on the given line.

(III) If a transversal is perpendicular to one of two parallel lines, it is perpendicular to the other also.

Our procedure has been to choose Statement I as a postulate (Postulate 22) and then, using that postulate and others, to prove Statement III (as Corollary 6-4-3). Instead of doing this, suppose that you accept Statement III as a "postulate." On the basis of this and the work before Section 6-5, prove Statement I.

Hint: Let m be a line and P a point not on m . Assuming that there are two lines, say ℓ_1 and ℓ_2 , each of which contains P and is parallel to m , try to reach a contradiction of Statement III.



*16. Make a list of different hypotheses pertaining to a quadrilateral, such that each of these hypotheses yields the conclusion that the quadrilateral is a parallelogram.

Hint: Obtain your information from theorems, or from discussion in the text or in class, or from problems.

Chapter 7

SIMILARITY

7-1. Introduction.

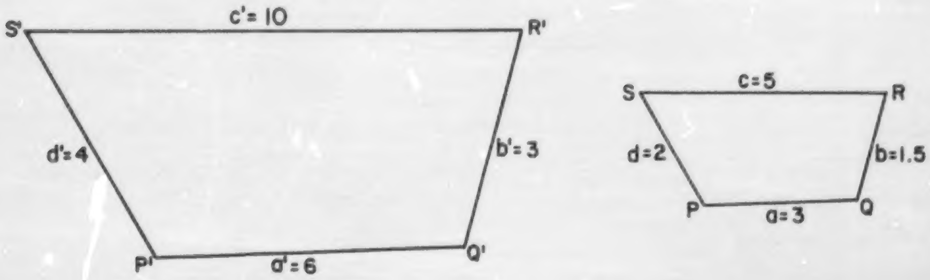
In Chapter 5 the idea of same size and shape in the world of physical things suggests the congruence concept in the world of formal geometry. In this chapter we will begin with the idea of same shape as it applies to material objects and then develop the mathematical concept of similarity. Consider a picture and a photographic enlargement of it. In the enlargement each portion of the picture has approximately the same shape as in the original, but not the same size.

Now consider a building and a floor plan for the building. The various rooms, corridors, doorways have approximately the same shape in the building as their representations have in the floor plan, but not the same size. There is a one-to-one correspondence between the corners in the building and the corners on the floor plan such that corresponding angles have approximately the same measure, but distances in the building are much larger than the corresponding distances on the plan. The plan has been drawn to scale. If the architect has chosen a scale in which one inch in the floor plan represents one foot in the building, then each distance in the building is approximately 12 times the corresponding distance on the plan. This relation between the corresponding distances is a basis for the concept of proportionality which we now develop.

393

7-2. Proportionality.

Note the lengths of the sides of the two polygons, as indicated in the figures below.



The two polygons are not congruent because each side of polygon P'Q'R'S' is twice as long as the corresponding side of polygon PQRS . One way to describe the relation between the two polygons is to say that the right-hand polygon is obtainable by shrinking the left-hand one, or the left-hand polygon is obtainable by stretching the one on the right.

Let us examine the correspondence, P'Q'R'S' ⟷ PQRS . It determines a one-to-one correspondence between sides; for instance, $\overline{PQ} \longleftrightarrow \overline{P'Q'}$, $\overline{QR} \longleftrightarrow \overline{Q'R'}$, ... It also establishes a one-to-one correspondence between the lengths of corresponding sides. We write this correspondence as follows:

$$(a', b', c', d') \longleftrightarrow (a, b, c, d)$$

with the understanding that $a' \longleftrightarrow a$, $b' \longleftrightarrow b$, $c' \longleftrightarrow c$, $d' \longleftrightarrow d$.

Notice that the lengths of the sides of the two quadrilaterals form two sequences of numbers, a', b', c', d', and a, b, c, d, standing in a special relation: each number in the first sequence is exactly twice the corresponding number in the second sequence. Then

$$a' = 2a,\ b' = 2b,\ c' = 2c,\ d' = 2d\ .$$

These equations give us a basis for saying that a', b', c', d' are proportional to a, b, c, d, and that 2 is the constant of proportionality. Another way of stating this relationship would be by writing the following equations:

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = \frac{d'}{d} = 2 .$$

In the following definition, we use the phrase "the numbers $q, r, s, \ldots$"; we do this because we do not wish to specify how many numbers are involved. There may be only two, in which case we mean "the numbers q, r." There may be three, in which case we mean "the numbers q, r, s." Or there may be four, or five, or a thousand, or more than that. In fact, we allow the possibility of infinitely many.

DEFINITIONS. Suppose there exists a one-to-one correspondence between the real numbers $q, r, s, \ldots$ and the real numbers $a, b, c, \ldots$ (that is, $q \longleftrightarrow a$, $r \longleftrightarrow b$, $s \longleftrightarrow c$, ...). The numbers $q, r, s, \ldots$ are said to be proportional to the numbers $a, b, c, \ldots$ if and only if there is a non-zero number k such that $q = ka$, $r = kb$, $s = kc$, ...

The number k is called the constant of proportionality.

Notation. We use "$\underset{p}{=}$" to mean "are proportional to." We also use $(a, b, c, \ldots)$ to represent an ordered sequence of numbers. For instance, "$(6, 12, 21) \underset{p}{=} (2, 4, 7)$" means $6, 12, 21$ are proportional to $2, 4, 7$, with $6 \longleftrightarrow 2$, $12 \longleftrightarrow 4$, and $21 \longleftrightarrow 7$.

In this example, we see that 3 is the constant of proportionality because $6 = 3 \cdot 2$, $12 = 3 \cdot 4$, $21 = 3 \cdot 7$.

In another example, $(5, 12, 13) \underset{p}{=} (15, 36, 39)$ means that 5, 12, 13 are proportional to 15, 36, 39, and thus we know that $5 = k \cdot 15$, $12 = k \cdot 36$, and $13 = k \cdot 39$. The constant of proportionality is k and $k = \frac{1}{3}$.

Problem Set 7-2a

1. Indicate whether each of the following statements is true or false according to the above definition. If true, give the proportionality constant k. (Part (a) is given as an example.)

   (a) $(6, 12) \underset{p}{=} (2, 4)$. True. $k = 3$

   (b) $(2,4) \underset{p}{=} (6,12)$.

   (c) $(2, 4, 6) \underset{p}{=} (12, 24, 36)$.

   (d) $(4, 6, -10, 0) \underset{p}{=} (2, 3, -5, 0)$.

   (e) $(2, 12, -8) \underset{p}{=} (.5, 3, -2)$.

   (f) $(4, 3) \underset{p}{=} (2, 6)$.

   (g) $(-6, -10, -2) \underset{p}{=} (9, 15, 3)$.

   (h) $(1, 4, 19, 20) \underset{p}{=} (3, 12, 37, 60)$.

   (i) $(3, 4, 5) \underset{p}{=} (6, 8, 10)$.

   (j) $(6, 8, 10) \underset{p}{=} (18, 24, 30)$.

   (k) $(3, 4, 5) \underset{p}{=} (18, 24, 30)$.

   (l) $(a, b, -c, 0) \underset{p}{=} (a, b, -c, 0)$.

   (m) $(4, 16) \underset{p}{=} (2, 4)$.

   (n) $(0, 0) \underset{p}{=} (2, -2)$.

   (o) $(0, 2) \underset{p}{=} (0, -2)$.

2. Complete each of the following to form a proportionality.

   (a) $(21, \_\_, \_\_, \_\_) \underset{p}{=} (7, 16, 3, 5)$. $k = \_\_$.

   (b) $(\_\_, \_\_, 1) \underset{p}{=} (15, 20, 5)$. $k = \_\_$.

   (c) $(\_\_, \_\_, \_\_, \_\_) \underset{p}{=} (6, 12, 0, -3)$. $k = 6$.

   (d) $(\frac{1}{2}, \frac{1}{3}) \underset{p}{=} (\frac{1}{6}, \_\_)$. $k = \_\_$.

   (e) $(5, 12, 13) \underset{p}{=} (\_\_, \_\_, \_\_)$. $k = \frac{1}{2}$.

3. If $(4, 3) \underset{p}{=} (6, x)$, one conclusion is that $4 = k \cdot 6$, and $3 = k \cdot x$. Which of the following are also correct conclusions?

   (a) $(3, 4) \underset{p}{=} (x, 6)$.

   (b) $(4, 6) \underset{p}{=} (3, x)$.

   (c) $\frac{4}{6} = \frac{3}{x}$.

   (d) $\frac{4}{3} = \frac{6}{x}$.

   (e) $\frac{3}{4} = \frac{x}{6}$.

   (f) $4 \cdot x = 3 \cdot 6$.

   (g) $(4, 4 + 3) \underset{p}{=} (6, 6 + x)$.

*4. Does $(a, b) \longleftrightarrow (c, d)$ indicate the same correspondence as $(b, a) \longleftrightarrow (d, c)$? If $(a, b) \underset{p}{=} (c, d)$ with proportionality constant k, is it true that $(b, a) \underset{p}{=} (d, c)$ with the same proportionality constant?

*5. When $(3, 4) \underset{p}{=} (6, 8)$, the proportionality constant is _____ . But if $(6, 8) \underset{p}{=} (3, 4)$, the proportionality constant is _____ . How are your two answers related?

*6. Since $(3, 5) \underset{p}{=} (9, 15)$ and $(9, 15) \underset{p}{=} (18, 30)$ is it true that $(3, 5) \underset{p}{=} (18, 30)$? Find the proportionality constant for each and try to express the relationship that exists.

*7. Suppose that $6 = 3 \cdot x$ and $12 = 3 \cdot y$.

(a) Is it true that $(6, 12) \underset{p}{=} (x, y)$?

(b) Is it true that $\frac{6}{x} = \frac{12}{y}$?

(c) Is it true that $6y = 12x$?

8. If $(c, d) \underset{p}{=} (a, b)$ and if $b = 0$, must $d = 0$? Why?

If $a \neq 0$, can $c = 0$?

*9. If $(c, 3) \underset{p}{=} (a, 6)$, and $a \neq 0$, is $\frac{a}{c} = \frac{6}{3}$? Is $\frac{c}{3} = \frac{a}{6}$?

*10. If $(a, b) \underset{p}{=} (2, 3)$, is $(a, 2) \underset{p}{=} (b, 3)$?

Is $(3, b) \underset{p}{=} (2, a)$?

*11. Using the definition of proportionality prove each of the following statements if a, b, c, d are positive numbers.

(a) If $(a, b) \underset{p}{=} (c, d)$, then $(b, a) \underset{p}{=} (d, c)$.

(b) If $(a, b) \underset{p}{=} (c, d)$, then $(a, c) \underset{p}{=} (b, d)$.

(c) If $(a, b) \underset{p}{=} (c, d)$, then $ad = bc$.

(d) If $ad = bc$, then $(a, b) \underset{p}{=} (c, d)$.

(Note: If $c = 0$, would Part (b) still be true? If $c = d = 0$ or if $a = b = 0$, would Part (d) still be true?)

12. If $(a, b) \underset{p}{=} (c, d)$ does it follow that

$(5a, b) \underset{p}{=} (5c, d)$?

How are their proportionality constants related?

13. Complete each of the following to form a proportionality.

(a) $(2, 4) \underset{p}{=} (4, \_\_)$.

(b) $(4, \_\_) \underset{p}{=} (6, 9)$.

(c) $(\_\_, 6) \underset{p}{=} (6, 12)$.

(d) $(\_\_, \_\_, \_\_, 3) \underset{p}{=} (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{3})$.

(e) $(2, \_\_, \_\_, \_\_) \underset{p}{=} (\sqrt{2}, 1, 2, \sqrt{8})$.

(f) $(\_\_, \_\_, \_\_, .8d) \underset{p}{=} (5a, 1.5b, .62c, 4d)$.

(g) $(\_\_, 3, \_\_, \_\_) \underset{p}{=} (1, \sqrt{3}, 3, 9)$.

(h) $(\_\_, 3) \underset{p}{=} (4, 1)$.

(i) $(\_\_, 4) \underset{p}{=} (4, 3)$.

(j) $(2, \sqrt{10}) \underset{p}{=} (\_\_, 5)$.

(k) $(\sqrt{3}, \sqrt{2}) \underset{p}{=} (9, \_\_)$.

(l) $(\_\_, 8, 6, \sqrt{2}) \underset{p}{=} (3, 4\sqrt{2}, \_\_, \_\_)$.

Properties of Proportionality.

You may already be familiar with some of the properties of proportionality from your study of algebra. As you might expect, we first note that a proportionality has the reflexive, symmetric, and transitive properties just as we found in the cases of equality and congruence.

1. The reflexive property of proportionality

$$(a, b, c, \ldots) \underset{p}{=} (a, b, c, \ldots)$$

The proportionality has one as its proportionality constant.

2. The symmetric property of proportionality

If $(a', b', c', \ldots) \underset{p}{=} (a, b, c, \ldots)$, then

$$(a, b, c, \ldots) \underset{p}{=} (a', b', c', \ldots) .$$

The hypothesis tells us that $a' = ka$, $b' = kb$, $c' = kc$, $\ldots$. Since $k \neq 0$, it follows that $a = \frac{1}{k}a'$, $b = \frac{1}{k}b'$, $c = \frac{1}{k}c'$, $\ldots$. Therefore, $(a, b, c, \ldots) \underset{p}{=} (a', b', c', \ldots)$ with $\frac{1}{k}$ as the constant of proportionality.

3. The transitive property of proportionality

If $(a, b, c, \ldots) \underset{p}{=} (e, f, g, \ldots)$ and

$(e, f, g, \ldots) \underset{p}{=} (q, r, s, \ldots)$, then

$(a, b, c, \ldots) \underset{p}{=} (q, r, s, \ldots)$.

If h and k are the respective constants of proportionality, then $a = he$ and $e = kq$; therefore $a = hkq$. Similarly $b = hkr$, $c = hks$, $\ldots$. Therefore, $(a, b, c, \ldots) \underset{p}{=} (q, r, s, \ldots)$.

4. The addition property of proportionality

If $(a, b, c, \ldots) \underset{p}{=} (q, r, s, \ldots)$, then $(a + b + c + \ldots, a, b, c, \ldots) \underset{p}{=} (q + r + s + \ldots, q, r, s, \ldots)$.

Because $a = kq$, $b = kr$, $c = ks$, $\ldots$, $a + b + c + \ldots = k(q + r + s + \ldots)$ and the property is proved.

As examples of the addition property consider the following:

Example 1. Given that $(a, b, c) \underset{p}{=} (3, 4, 5)$, Show that $(a + b + c, a, b, c) \underset{p}{=} (12, 3, 4, 5)$.

This follows from the addition property since $12 = 3 + 4 + 5$.

Example 2. Given: $(r, s, t) \underset{p}{=} (e, f, g)$. Write a proportionality that starts with (1) $r + s$, (2) r and has $r + s$ as second member of the first sequence.

(1) $(r + s, r, s, t) \underset{p}{=} (e + f, e, f, g)$ by the addition property.

(2) $(r, r + s, t, s) = (e, e + f, g, f)$ from (1) since this is the same correspondence as in (1) .

We frequently work with proportionalities whose sequences consist of only two numbers. Such proportionalities are generally called proportions and have special properties which we list below.

> DEFINITION. A proportionality is a proportion if and only if each of its sequences consists of two numbers.

Properties of proportion.

Given that a, b, c, d are positive numbers then:

1. (Inversion Property of Proportion). If $(a, b) \underset{p}{=} (c, d)$, then $(b, a) \underset{p}{=} (d, c)$.
2. (Alternation Property of Proportion). If $(a, b) \underset{p}{=} (c, d)$, then $(a, c) \underset{p}{=} (b, d)$ or $(d, b) \underset{p}{=} (c, a)$.
3. (Product Property of Proportion). $(a, b) \underset{p}{=} (c, d)$ if and only if $ad = bc$.

In geometric applications of these properties of proportion the numbers a, b, c, d usually arise as lengths of segments. Therefore, there is no need for us to consider cases in which a, b, c, d are not all positive.

Example 3. Find x if x is positive and $(3, 7) \underset{p}{=} (x, 4)$.

By the Product Property, $3 \cdot 4 = 7 \cdot x$ and $x = \frac{12}{7}$.

Example 4. Given: y and z are positive and $4y = 7z$. Write three proportions starting with 4.

By the Product Property: $(4, 7) \underset{p}{=} (z, y)$.

By the Alternation Property: $(4, z) \underset{p}{=} (7, y)$.

By the Addition Property: $(4, 11) \underset{p}{=} (z, z + y)$.

These are some of the proportions that might start with 4.

Problem Set 7-2b

1. Is each of the following true or false? Why?

   (a) $(x, y) \underset{p}{=} (x, y)$.

   (b) If x, y, r, s are positive and $(x, y) \underset{p}{=} (r, s)$ with proportionality constant $\frac{2}{3}$, then $(s, r) \underset{p}{=} (y, x)$ with proportionality constant $\frac{3}{2}$.

   (c) If x, y are positive and $(2, 3) \underset{p}{=} (x, y)$, then $(x, y) \underset{p}{=} (3, 2)$.

   (d) If $(m, n) \underset{p}{=} (r, s)$ with $k = 2$ and
   if $(m, n) \underset{p}{=} (b, c)$ with $k = 6$ then
   $(r, s) \underset{p}{=} (b, c)$ with $k = 12$.

   (e) If $(2, 3) \underset{p}{=} (a, b)$ with $k = 3$, then
   $(b, a) \underset{p}{=} (3, 2)$ with $k = \frac{1}{3}$.

   (f) If $(2, 3) \underset{p}{=} (a, b)$, then
   $(2, 3, 5) \underset{p}{=} (a, b, a + b)$.

   (g) If x is positive and $(2, 3) \underset{p}{=} (x, 6)$, then $(3, 2) \underset{p}{=} (6, x)$.

   (h) If x, y are positive and $(x, y) \underset{p}{=} (2, 3)$, then $2x = 3y$.

2. Given: $(2, 7, 8) \underset{p}{=} (x, y, z)$ and x, y, z positive. Write a proportion that starts with (a) 7, (b) z, (c) 10, (d) 6 .

3. Given: $5x = 2y$, x and y positive. Write three proportions starting with 5 .

4. Find the positive number x in each of the following proportions:

(a) $(3, 6) \underset{p}{=} (2, x)$.

(b) $(3, 5) \underset{p}{=} (4, x)$.

(c) $(\frac{1}{2}, 3) \underset{p}{=} (x, 8)$.

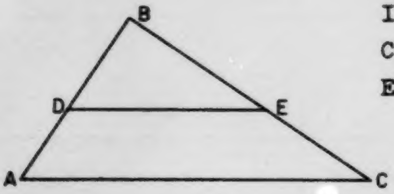
(d) $(4, 6) \underset{p}{=} (6, x)$.

(e) $(2, x) \underset{p}{=} (x, 8)$.

(f) $(2, x) \underset{p}{=} (x, 18)$.

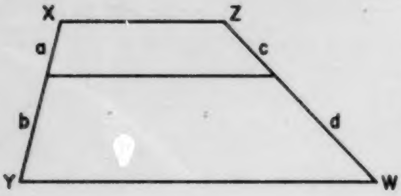
(g) $(1, x) \underset{p}{=} (x, 9)$.

(h) $(1, x) \underset{p}{=} (x, 2)$.

5. 

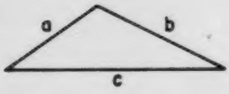
If AD, DB are proportional to CE, EB and if $AD = 3$, $DB = 4$, $EB = 6$, find CE .

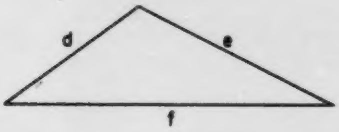
6. D is between A and B . E is between B and C . If $(AD, DB) \underset{p}{=} (CE, EB)$, prove $(AD, DB, AB) \underset{p}{=} (CE, EB, CB)$.

7. 

Let a, b, c, d represent the lengths of the segments pictured in the diagram. If $(a, b) \underset{p}{=} (c, d)$ and if $XY = 6$, $ZW = 8$, $a = 2$, find b, c, d .

8. Let a, b, c, d, e, f be lengths of the segments. If $(a, b, c) \underset{p}{=} (d, e, f)$, and if $a = 3$, $b = 4$, $c = 6$, $f = 9$, find d and e .

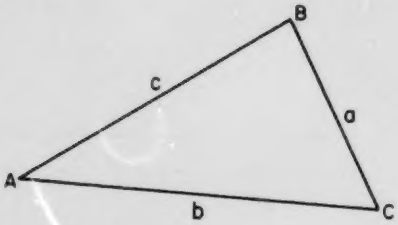


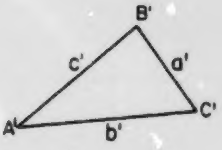


7-3. Similarities Between Polygons.

In this section we present a definition of a similarity between two polgyons.

Let us first consider two triangles, as shown in the diagram.





Note that the figure indicates that the side of triangle ABC opposite $\angle A$ has length a, the side opposite $\angle B$ has length b, and the side opposite $\angle C$ has length c. Likewise for $\triangle A'B'C'$. Suppose now that the correspondence

$$ABC \longleftrightarrow A'B'C'$$

between the triangles has the properties that corresponding angles are congruent and that the measures of the sides of one triangle are proportional to the measures of the corresponding sides of the other. The latter requirement means that there is a non-zero number k such that

$$a = ka' , \quad b = kb' , \quad c = kc' .$$

A correspondence which has these properties is called a similarity, and we write

$$\triangle ABC \sim \triangle A'B'C' ,$$

a statement which we read as "triangle ABC is similar to triangle A'B'C' ."

Try to formulate a description of the notion of a similarity between two quadrilaterals. What do you think it means to say that one hexagon is similar to another hexagon?

DEFINITION. Let n be a natural number. A one-to-one correspondence between the vertices of two convex polygons, each with n sides, (or between the vertices of a convex polygon and themselves) such that corresponding angles are congruent and such that the measures of corresponding sides are proportional is called a *similarity*, and the two polygons are said to be *similar to* one another.

Notation. If $A_1A_2...A_n$ and $B_1B_2...B_n$ are polygons with n sides, then the statement that "the correspondence $A_1A_2...A_n \longleftrightarrow B_1B_2...B_n$ is a similarity" is written:

$$A_1A_2...A_n \sim B_1B_2...B_n .$$

Note that the definition imposes three requirements for polygons to be similar: (1) that a particular one-to-one correspondence between vertices be specified; (2) that corresponding angles be congruent; (3) that the measures of corresponding sides be proportional. Let us examine these requirements more closely.

First consider the correspondence $ABC \longleftrightarrow A'B'C'$ between certain vertices of the polygons shown in Figure a. Under the correspondence $ABC \longleftrightarrow A'B'C'$, corresponding angles are congruent and the measures of corresponding sides are proportional. In spite of this, the correspondence is not a one-to-one correspondence betwen the set of *all* vertices of one polygon and the set of *all* vertices of the other polygon. In fact, these two polygons cannot be similar, since there can be no one-to-one correspondence between the vertices of a quadrilateral and the vertices of a pentagon.

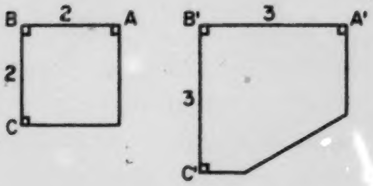


Figure a

Second, consider the correspondence $ABCD \longleftrightarrow A'B'C'D'$ between the vertices of the two quadrilaterals shown in Figure b. The measures of corresponding sides are proportional, but the correspondence is not a similarity. Which requirement is not satisfied?

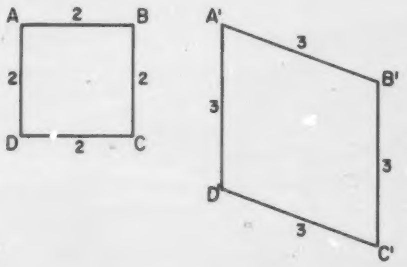


Figure b

Third, consider the correspondence $PQRS \longleftrightarrow UVWX$ between the vertices of the two quadrilaterals in Figure c. The corresponding angles are congruent, but the parallelograms are not similar. Why?

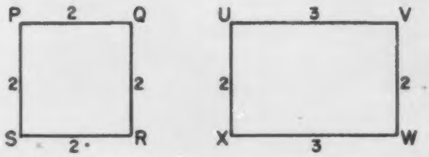


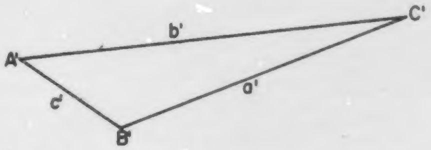
Figure c

Fourth, a correspondence between the vertices of the quadrilateral EFGH and the vertices of the quadrilateral KLMN is given by: $E \longleftrightarrow K$, $F \longleftrightarrow M$, $G \longleftrightarrow L$, $H \longleftrightarrow N$. Explain why this correspondence cannot be a similarity.

Fifth, let ABC be a right triangle with right angle at A . The correspondence $ABC \longleftrightarrow BCA$ is not a similarity. Why? However, there is another correspondence between the vertices which shows that $\triangle ABC$ is similar to itself; what is this correspondence?

Suppose we have two triangles and a similarity between them: $\triangle ABC \sim \triangle A'B'C'$ with a proportionality constant h .

Then $\angle A \cong \angle A'$
$\angle B \cong \angle B'$
$\angle C \cong \angle C'$,

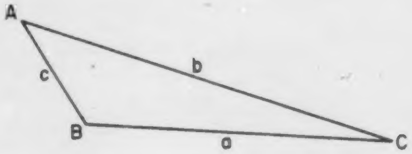


and there is a positive number h such that

$a = ha'$,

$b = hb'$,

$c = hc'$.



What familiar situation occurs if $h = 1$? Then $a = a'$, $b = b'$, $c = c'$. This means that corresponding sides of the triangles are _____ . (Fill the blank with an appropriate word.) Since also the corresponding angles are congruent, the correspondence

$$ABC \leftrightarrow A'B'C'$$

is a _____ between the triangles. (Fill in the blank.)

Conversely, if the correspondence $ABC \leftrightarrow A'B'C'$ is a congruence, then corresponding angles are congruent and the measures of corresponding sides are proportional, the constant of proportionality being the number 1 . We thus see that our previous description of congruence for triangles, as given in Chapter 5, agrees with the following definition for polygons:

> DEFINITION. A similarity between two convex polygons is called a congruence if and only if the constant of proportionality for the measures of corresponding sides is 1 .

We use the symbol $\cong$ to denote "is congruent to" for polygons, just as we have for triangles.

Congruence is a special case of similarity. If a correspondence between convex polygons is a congruence, then it is also a similarity; the converse of course is not true. During the remaining sections of this chapter, we study similarity between triangles; in our later work we investigate more fully similarity and congruence between polygons with more than three sides.

First, however, we state the following general properties.

THEOREM 7-1. The relation of similarity between convex polygons is reflexive, symmetric, and transitive.

Suppose P, Q, R are any convex polygons, each having the same number of sides. Our theorem says:

(a) $P \sim P$.
(b) If $P \sim Q$, then $Q \sim P$.
(c) If $P \sim Q$ and $Q \sim R$, then $P \sim R$.

We show a proof for the symmetric property as it applies to triangles. It could be easily modified to apply to any two polygons having the same number of sides.

$$\text{If } \triangle ABC \sim \triangle DEF \text{ , then } \triangle DEF \sim \triangle ABC \text{ .}$$

The definition of a similarity tells us:
$(AB, BC, CA) \underset{p}{=} (DE, EF, FD)$ and $\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$.
By the symmetric properties of proportionality and angle congruence, we can say,
$(DE, EF, FD) \underset{p}{=} (AB, BC, CA)$ and $\angle D \cong \angle A$, $\angle E \cong \angle B$, $\angle F \cong \angle C$.
Thus, we conclude that $\triangle DEF \sim \triangle ABC$.

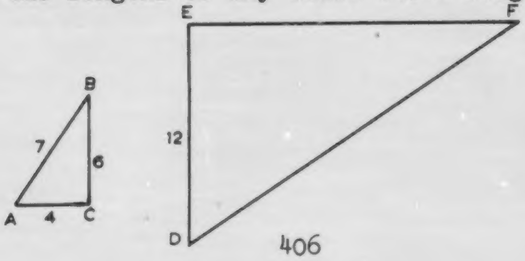
We leave the proof for the transitive property of triangle similarity as a problem in the next problem set.

Problem Set 7-3

1. In a certain floor plan, the scale tells us that $\frac{1}{4}$" represents one foot. What measures would be represented by 1" , $\frac{1}{2}$" , 6" , $\frac{1}{8}$" ? What would be the dimensions in the drawing of a floor which is 12' x 20' ?

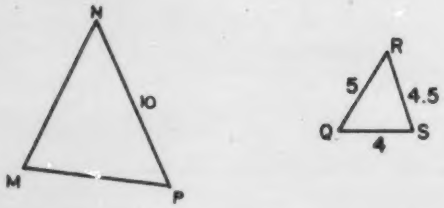
2. In each of the following pairs of similar triangles, consider the lengths of the sides of the first named triangle proportional to the lengths of the sides of the second and give the constant of proportionality. Also, give the lengths of any sides whose lengths are not given.

(a)



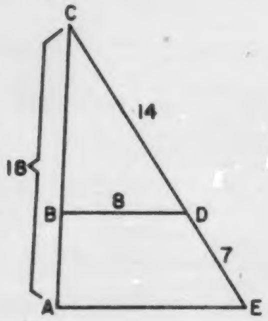
$\triangle ABC \sim \triangle DFE$

(b)



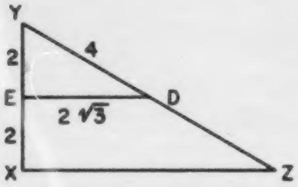
$\triangle MNP \sim \triangle SRQ$

(c)



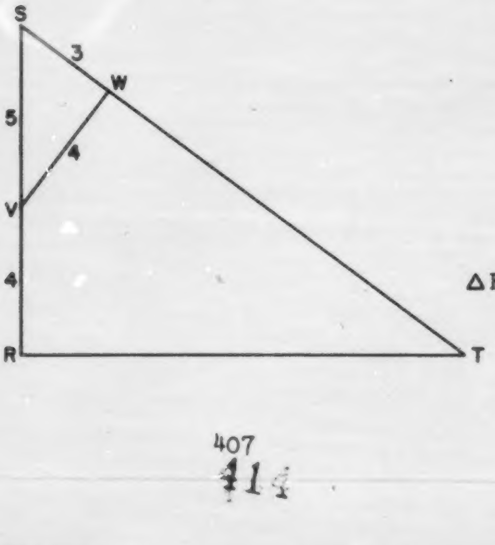
$\triangle ACE \sim \triangle BCD$

(d)



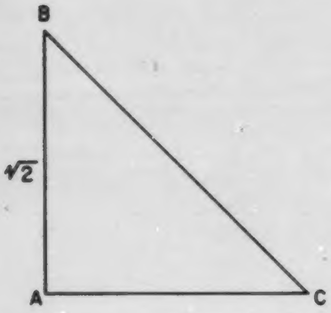
$\triangle XYZ \sim \triangle EYD$

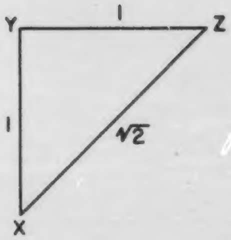
(e)



$\triangle RST \sim \triangle WSV$

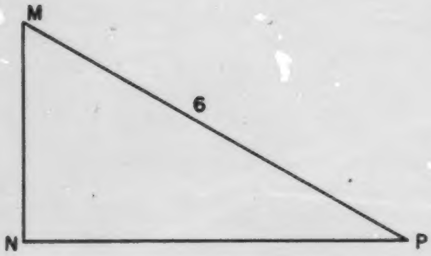
(f)

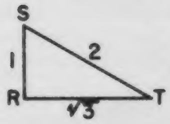




$\triangle ABC \sim \triangle YXZ$

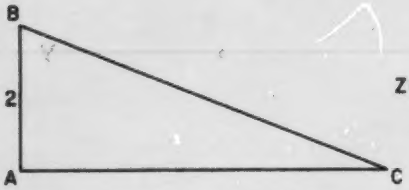
(g)

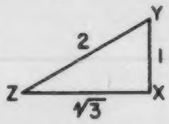


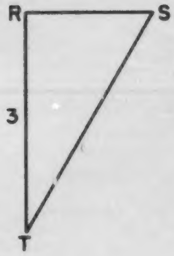


$\triangle MNP \sim \triangle SRT$

3. Given that $\triangle ABC \sim \triangle XYZ \sim \triangle RST$.







(a) Find BC, AC, RS, ST .

(b) Give the proportionality constant for

(1) $\triangle ABC \sim \triangle XYZ$.
(2) $\triangle XYZ \sim \triangle ABC$.
(3) $\triangle ABC \sim \triangle RST$.
(4) $\triangle XYZ \sim \triangle RST$.
(5) $\triangle RST \sim \triangle RST$.

4. If $\triangle ABC \sim \triangle A'B'C'$ with a proportionality constant 5 and if $\triangle A'B'C' \sim \triangle A''B''C''$ with a proportionality constant $\frac{1}{5}$, then $\triangle ABC \sim \triangle A''B''C''$ with a proportionality constant ___?___ .

5

5. If polygon P is similar to polygon P' with a proportionality constant 4 and P' is similar to P'' with a proportionality constant 2, then polygon P is similar to polygon P'' with a proportionality constant ___?___ .

6. If $\triangle ABC \sim \triangle XYZ$ with a proportionality constant $\sqrt{3}$, then $\triangle XYZ \sim \triangle ABC$ with a proportionality constant ___?___ .

7. A triangle has sides 6, 8, 4. The shortest side of a similar triangle is 2. What is the constant of proportionality? What is the ratio of a pair of corresponding sides? What is the ratio of the perimeter of the first triangle to the perimeter of the second triangle?

8. Quadrilateral A has sides of length 6, 9, 12, 11. The shortest side of a similar quadrilateral, B, has length 18. What is the length of each of the other sides of B? What is the relation between each angle of A and the corresponding angle of B?

9. The perimeter, (that is, the sum of the lengths of the sides) of a triangle is 48. Its shortest side has length 11. What is the length of the shortest side of a similar triangle whose perimeter is 144? What is the relation of each angle of the given triangle to the corresponding angle of the other triangle?

10. Prove Theorem 7-1 (c) for triangles only.

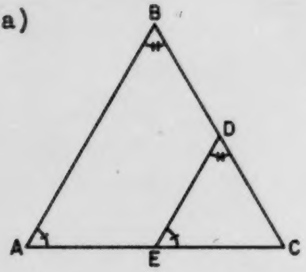
11. If $\triangle ABC \sim \triangle XYZ$ and $\triangle XYZ \cong \triangle MNP$, then what relationship exists between $\triangle ABC$ and $\triangle MNP$?

12. If $\triangle ABC \sim \triangle XYZ$, and $\triangle XYZ \cong \triangle ZYX$, then if $AB = 8$, and $AC = 5$, can we determine BC?

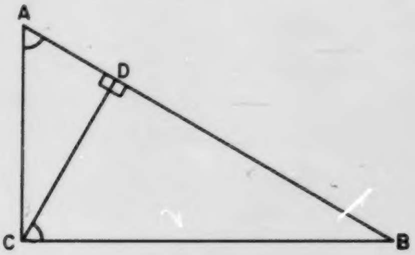
13. If $\triangle ABC \sim \triangle XYZ$ and $AB = XY$, must $\triangle ABC$ be congruent to $\triangle XYZ$?

14. In each of the following diagrams the two distinct triangles indicated are similar. In (b) the triangles are $\triangle ACD$ and $\triangle BDC$. On the basis of the given information determine a correspondence between the two triangles which is a similarity.

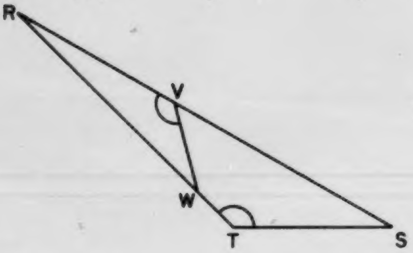
(a)



(b)



(c)



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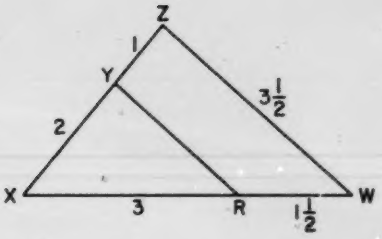

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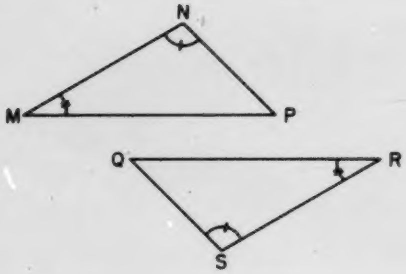

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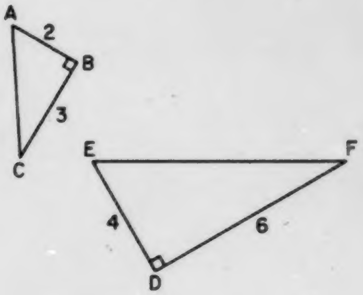

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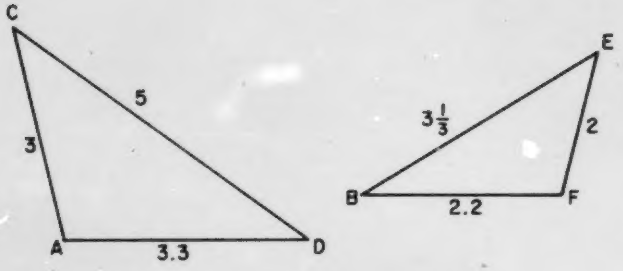
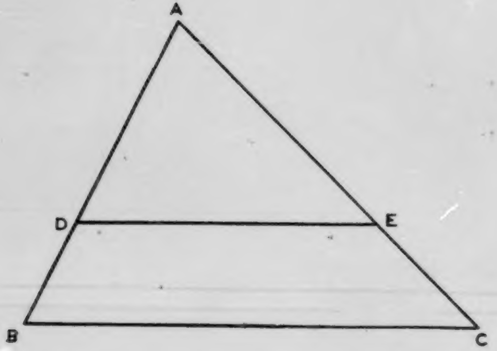


7-4. An Experiment and a Postulate.

The purpose of this experiment is to make our next postulate plausible. We will use the postulate to develop formally the concept of similarity.

Experiment. Draw a triangle ABC (not isosceles) with $\overleftrightarrow{DE}$ parallel to $\overleftrightarrow{BC}$ and intersecting the interiors of the other two sides, as shown below, in D and E .



Using a ruler with a scale in centimeters, find AB, AD, DB . Then find AC, AE, EC .

Using your measures, test whether

$$(AB, AD, DB) \underset{p}{=} (AC, AE, EC) .$$

Consider this one-to-one correspondence between two sets of points:

$$ADB \longleftrightarrow AEC .$$

This correspondence establishes the following correspondences

$$\overline{AB} \longleftrightarrow \overline{AC} , \overline{AD} \longleftrightarrow \overline{AE} , \overline{DB} \longleftrightarrow \overline{EC} .$$

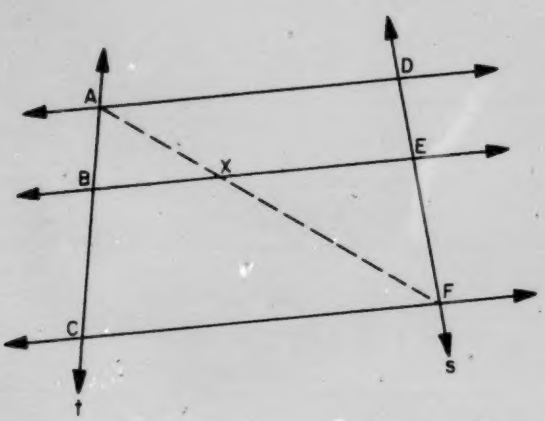
The results of this experiment suggest the following postulate.

Postulate 23. (The Proportional Segments Postulate) If a line is parallel to one side of a triangle and intersects the other two sides in interior points, then the measures of one of those sides and the two segments into which it is cut are proportional to the measures of the three corresponding segments in the other side.

We use this postulate in the following theorem to develop a property of proportionality and later to develop properties of similarity. First, however, let us agree to shorten a certain phrase which appears frequently. Suppose we have a one-to-one correspondence between two sets of segments. If the measures of the segments in one set are proportional to the measures of the segments in the other set, we often express this by saying that "corresponding segments are proportional." When we mean that the measures of the sides of one triangle are proportional to the measures of the corresponding sides of the other triangle, we could say, "corresponding sides are proportional."

THEOREM 7-2. Given three coplanar parallel lines and two transversals, the corresponding segments on the transversals are proportional.

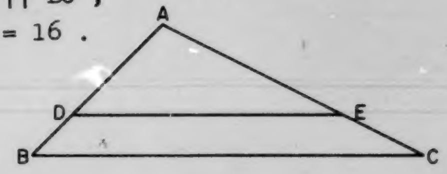
Proof: Let the transversals be t and s. Let the parallel lines be $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, $\overleftrightarrow{CF}$, where A, B, C are the intersections of these lines with t, and D, E, F are the intersections with s. Consider segment $\overline{AF}$. We need to prove that the numbers AB, BC, AC are proportional to the numbers DE, EF, DF. The proof is left as a problem in the next problem set.



Hint: In this diagram, show (AB, BC) $\underset{p}{\equiv}$ (AX, XF) $\underset{p}{\equiv}$ (DE, EF) .

Problem Set 7-4

1. (a) In the figure, if $\overline{DE} \parallel \overline{BC}$, AD = 7 , AB = 10 , AC = 16 . What is AE ?

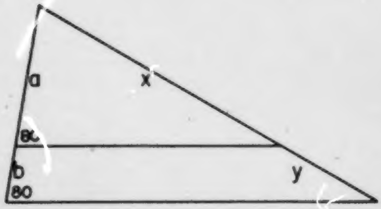


(b) Fill in the blanks below:

	AD	DB	AB	AE	EC	AC
(1)	3	3	___	4	___	___
(2)	___	2	6	___	4	___
(3)	3	___	7	___	5	___
(4)	2	3	___	___	___	7
(5)	2	1	___	___	2	___

(c) Are AD, AE, proportional to AC, AB ?

2. In this figure, the lengths of the segments are a, b, x, y, and the measures of two angles are 80 , as indicated. Complete the following:



(a) a, ___ are proportional to x, y .

(b) $(a + b)$, a are proportional to ___, x .

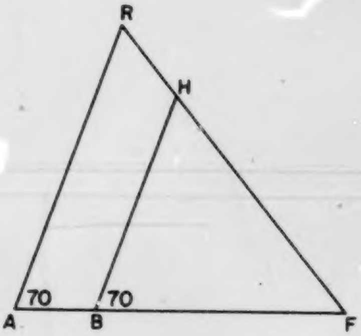
(c) a, x are proportional to ___, y .

(d) If $a + b = k(x +$ ___$)$, then $b = k$ ___ .

(e) If $a + b = h$ ___ then $x + y = hx$.

(f) $(a + b)$, $(x + y)$ are proportional to ___, y .

3. In this figure, find the missing lengths.



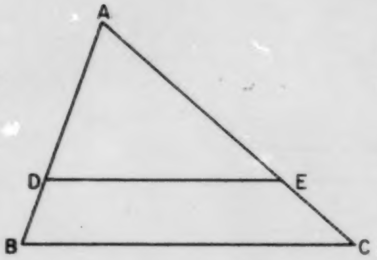
(a) If $RH = 4$, $HF = 7$, and $BF = 10$, then $AB =$ ___ .

(b) If $RH = 6$, $HF = 10$, and $AB = 3$, then $BF =$ ___ .

(c) If $RH = 5$, $RF = 20$, and $AF = 18$, then $BF =$ ___ .

4. Complete the proof in paragraph form for Theorem 7-2.

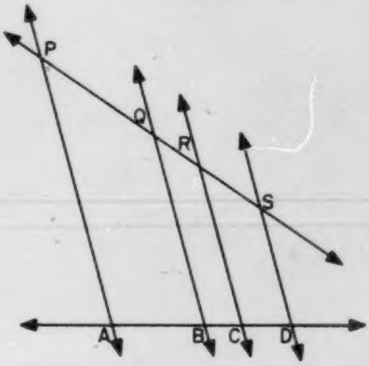
5. The converse of the Proportional Segments Postulate, although true, is not needed as a theorem in our formal development of geometry. The proof is interesting, however. Complete the proof outlined below and then write the converse of the Proportional Segments Postulate.



Hypothesis: Triangle ABC has line $\overleftrightarrow{DE}$ intersecting the interiors of $\overline{AB}$ and of $\overline{AC}$ respectively in D and E . Also, AB, AD are proportional to AC, AE . Prove that $\overleftrightarrow{DE}$ is parallel to $\overline{BC}$. (Hint: Consider a line through B parallel to $\overline{DE}$ and intersecting $\overrightarrow{AC}$ at F . Then prove that AB, AD are proportional to AF, AE ; and $\overline{AC} \cong \overline{AF}$, and C = F .)

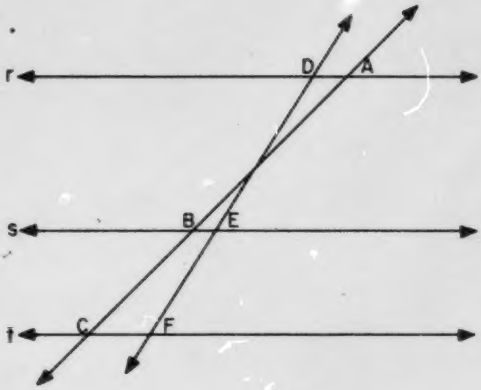
6. In this figure, lines $\overleftrightarrow{PA}$, $\overleftrightarrow{QB}$, $\overleftrightarrow{RC}$, and $\overleftrightarrow{SD}$ are parallel. AB = 5 , QR = 3 , BC = 2 , and RS = 4 .

   Find PQ and CD .



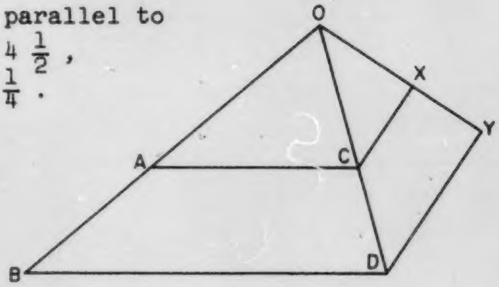
7. In this figure, lines r, s, and t are parallel. AB = 6 , BC = 4 , and DE = 5 .

   What is DF ?

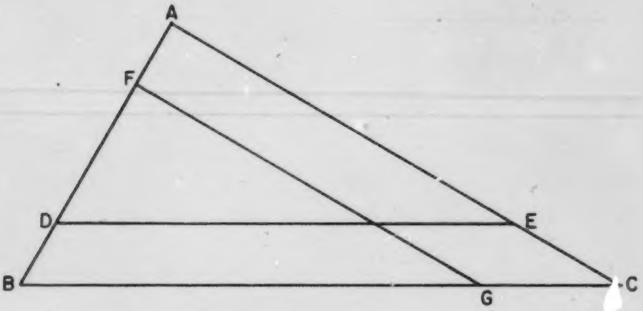


8. In this figure $\overline{AC}$ is parallel to $\overline{BD}$ and $\overline{CX}$ is parallel to $\overline{DY}$. $OA = 6$, $AB = 4\frac{1}{2}$, $OC = 4$ and $XY = 2\frac{1}{4}$.

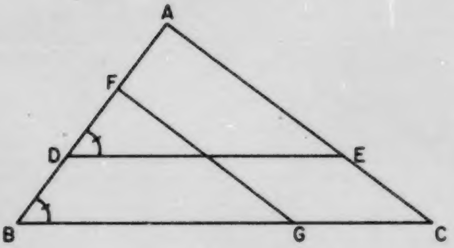
Find CD and OY .



*9. In this figure $\overleftrightarrow{FG}$ and $\overleftrightarrow{AC}$ are parallel, $\overleftrightarrow{DE}$ and $\overleftrightarrow{BC}$ are parallel, and $BF = AD$; prove $\triangle ADE \cong \triangle FBG$.



*10. $\overline{FG} \parallel \overline{AC}$.
$\overline{DE} \parallel \overline{BC}$.
$AB = 9$, $AC = 12$,
$BC = 15$. $AD = \frac{2}{3} AB$.
$BF = \frac{2}{3} AB$.



Problem: (1) Can we find DE ? (2) Is $\triangle ADE \sim \triangle ABC$?

Steps to aid in the solution:

(a) $AD = ?$
(b) AE is what part of AC ? Why? $AE = ?$
(c) $BF = \frac{2}{3} AB$. Does $BF = AD$?
(d) BG is what part of BC ? Why? $BG = ?$
(e) Is $\triangle ADE \cong \triangle FBG$? Why?
(f) Is $\overline{BG} \cong \overline{DE}$? $DE = ?$
(g) Are the sides of $\triangle ADE$ proportional to the sides of $\triangle ABC$?
(h) What else must we prove before we can say that $\triangle ADE \sim \triangle ABC$?

11. Suppose that $\overrightarrow{AC}$ and $\overline{AB}$ are given and that $\overrightarrow{AC}$ is not in $\overleftrightarrow{AB}$. If points P, Q, R in that order on $\overrightarrow{AC}$ are such that $AP = PQ = QR$, show that the lines through P and Q and parallel to $\overleftrightarrow{RB}$ trisect $\overline{AB}$.

7-5. Triangle Similarity Theorems.

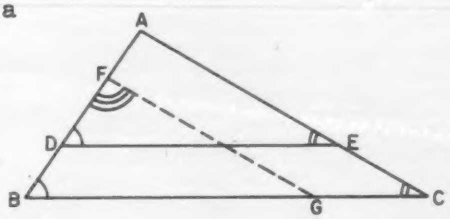
We conduct our formal study of similarity of triangles in a manner similar to that of congruence. You recall that we used six congruence conditions (three for sides and three for angles) to define a congruence between triangles. Then by congruence postulates and theorems, we were able to show that some of these six conditions implied the others. The application of these results reduced our work whenever we wished to prove two triangles congruent.

We are now going to find ways to prove two triangles are similar, ways that reduce the work demanded by the definition of similarity. These methods are provided by Theorems 7-4, 7-5, 7-6. As an aid in the proofs of these three theorems, we first prove that, given any triangle, there exists another triangle similar to it.

THEOREM 7-3. If a triangle and a positive number k are given, there is a triangle which is similar to the given triangle with proportionality constant k.

Proof: Let the given triangle be ABC with $a = BC$, $b = AC$, and $c = AB$.
We consider three cases:

(1) $k < 1$, (2) $k = 1$, (3) $k > 1$.



Case (1). If $k < 1$, there is a point D in $\overline{AB}$ such that $AD = kc$. The line which contains D and is parallel to $\overleftrightarrow{BC}$ intersects $\overline{AC}$, say in E. Then $AE = kb$. Why?

There is also a point F in $\overline{AB}$ such that $FB = kc$. A line containing F parallel to $\overleftrightarrow{AC}$ intersects $\overline{BC}$ in G and $BG = ka$.

Now, consider $ADE \longleftrightarrow FBG$, in which we know $AD = kc = FB$, $\angle B \cong \angle FDE$ (Why?) and $\angle A \cong \angle GFB$. Therefore $\triangle ADE \cong \triangle FBG$ (A.S.A.) and $DE = BG = ka$. We also note that $\angle C \cong \angle AED$ (Why?).

Now, consider $ADE \longleftrightarrow ABC$; we have $(AD, DE, EA) \underset{p}{=} (AB, BC, CA)$ with k as the constant of proportionality and we also know that all three pairs of corresponding angles are congruent. Therefore, by the definition of similarity we know that $\triangle ADE \sim \triangle ABC$.

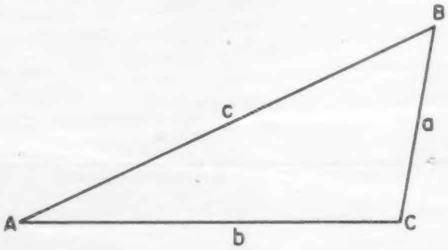
Case (2). If $k = 1$, then $D = B$ and $E = C$ and again $\triangle ADE \sim \triangle ABC$.

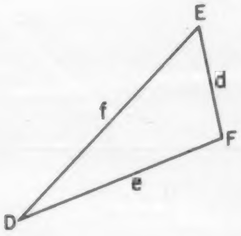
Case (3). If $k > 1$, then a slightly modified version of the argument given for Case (1) is needed. We would choose a point D on $\overrightarrow{AB}$ instead of $\overline{AB}$. The point F would lie on $\overrightarrow{BA}$ such that A was between B and F. Using this new diagram, you would find that the proof for Case (1) would then, with the changes prescribed, also prove in Case (3) that a triangle exists which is similar to the given triangle with the proportionality constant k.

Experiments

In each of these experiments, consider triangles ABC and DEF and the correspondence $ABC \longleftrightarrow DEF$.

1. Draw triangles ABC and DEF such that the corresponding sides are proportional.

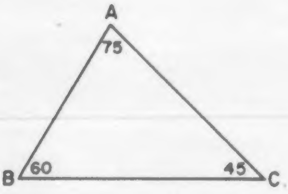


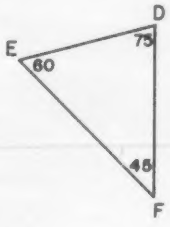


$$\begin{cases} a = kd \\ b = ke \\ c = kf \end{cases}$$

Measure the corresponding angles. Do the corresponding angles seem to be congruent? Do the triangles seem to be similar?

2. Draw triangles ABC and DEF such that the corresponding angles are congruent. (For instance, you may choose the measures of the angles to be 75, 60, 45, as in the diagram.)





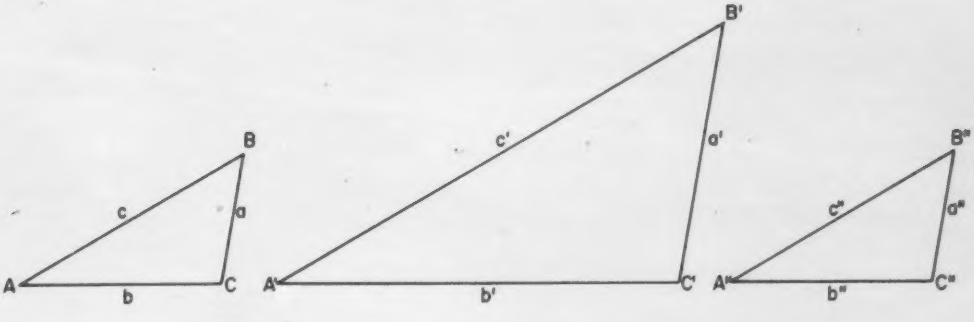
Measure the sides of the triangles. Do the corresponding sides seem to be proportional? Do the triangles seem to be similar?

3. Draw triangles ABC and DEF such that some other combination of conditions is satisfied--such as, just one pair of corresponding angles are congruent, or two pairs of corresponding angles are congruent, or two sides of one triangle are proportional to the corresponding sides of the other, or some combination of these or other conditions. Try to select conditions which seem to guarantee that the correspondence $ABC \longleftrightarrow DEF$ is a similarity. If you succeed, what conditions did you choose?

THEOREM 7-4. (The S.S.S. Similarity Theorem) A correspondence between two triangles such that corresponding sides are proportional is a similarity.

Proof: Hypothesis: $ABC \longleftrightarrow A'B'C'$ is a correspondence between triangles ABC and A'B'C'. There is a positive number k such that $a = ka'$, $b = kb'$, $c = kc'$.

To prove: $\triangle ABC \sim \triangle A'B'C'$.



Statements	Reasons
1. There is a triangle $A''B''C''$ similar to $A'B'C'$ with proportionality constant k.	1. Theorem 7-3.
2. $a'' = ka'$, $b'' = kb'$, $c'' = kc'$.	2. Corresponding sides of similar triangles are proportional.
3. But $ka' = a$, $kb' = b$, $kc' = c$.	3. Hypothesis.
4. $\overline{BC} \cong \overline{B''C''}$, $\overline{CA} \cong \overline{C''A''}$, $\overline{AB} \cong \overline{A''B''}$.	4. Segments with equal measures are congruent.
5. $\triangle ABC \cong \triangle A''B''C''$.	5. S.S.S. Congruence Postulate.
6. $\triangle ABC \sim \triangle A'B'C'$.	6. Transitive property of polygon similarity.

THEOREM 7-5. (The S.A.S. Similarity Theorem) If a correspondence between two triangles has the properties that two sides of one triangle are proportional to the corresponding sides of the other and that the included angles are congruent, then the correspondence is a similarity.

Proof: Hypothesis: $ABC \longleftrightarrow A'B'C'$

$AB = kA'B'$, $BC = kB'C'$, $\angle B \cong \angle B'$.

There is a $\triangle A''B''C''$ which is similar to $\triangle A'B'C'$ with proportionality constant k.

Therefore $A''B'' = kA'B' = AB$, $\angle B'' \cong \angle B' \cong \angle B$, $B''C'' = kB'C' = BC$ and $\triangle ABC \cong \triangle A''B''C'' \sim \triangle A'B'C'$. Therefore $\triangle ABC \sim \triangle A'B'C'$.

The problem of writing this as a two-column proof is left to you in the next problem set.

THEOREM 7-6. (The A.A. Similarity Theorem) If a correspondence between two triangles has the property that two angles of one triangle are congruent to the corresponding angles of the other, then the correspondence is a similarity.

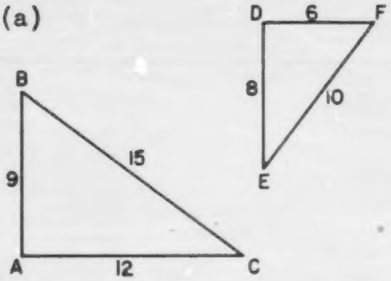
Let the correspondence between the two triangles be $ABC \leftrightarrow A'B'C'$ and $\angle A \cong \angle A'$, $\angle B \cong \angle B'$. There is a positive number k such that $AB = kA'B'$. Now, consider $\triangle A''B''C'' \sim \triangle A'B'C'$ with k as proportionality constant.

With this start as a hint, it is left to you to write the complete proof as a problem in the next set.

Problem Set 7-5

1. For every pair of triangles which can be proved similar, state the Similarity Theorem which would be used and state the one-to-one correspondence between the vertices of the triangles which is a similarity.

(a)



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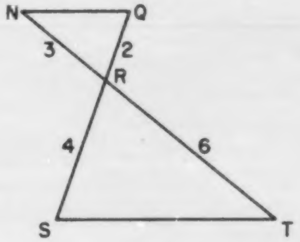

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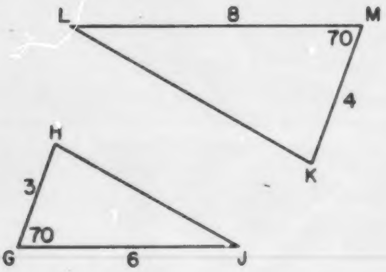
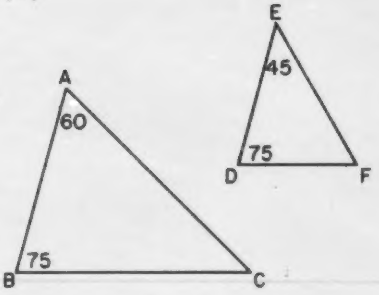


(d)



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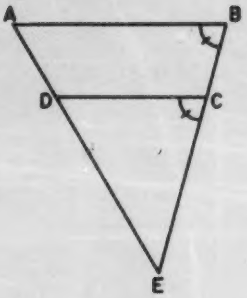

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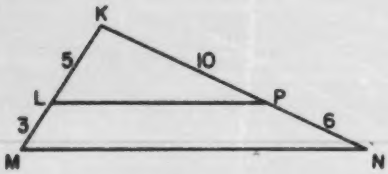

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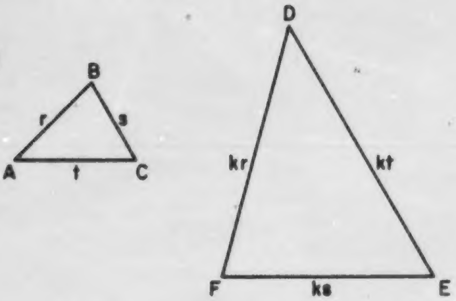

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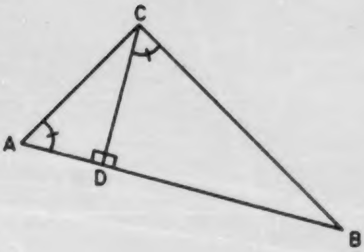
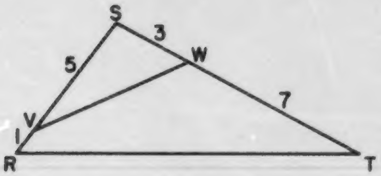


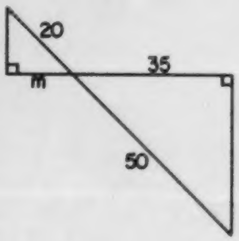
(i)



2. Prove Theorem 7-5 in a two-column proof.
3. Prove Theorem 7-6 in a two-column proof.

4. In each of the following, determine the length m if there is sufficient information given. In the diagrams, all points are coplanar and also collinear as indicated.

(a)



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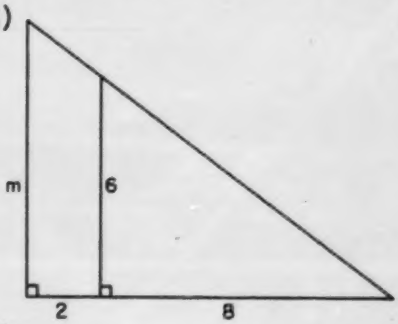

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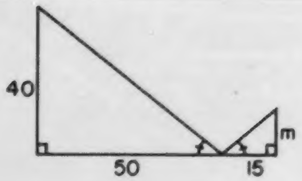

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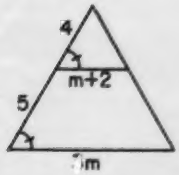

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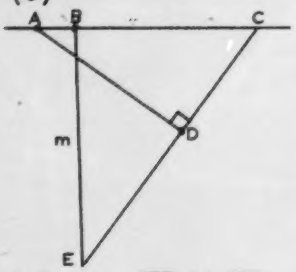

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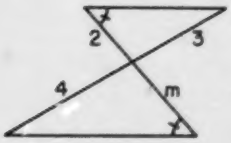
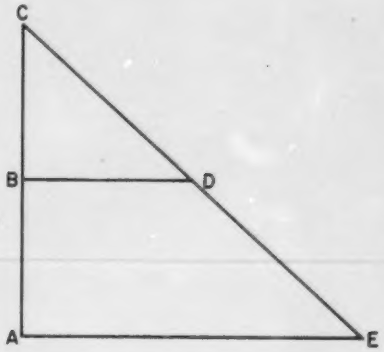


5. (a) If $\overline{BD} \parallel \overline{AE}$, prove $\triangle ACE \sim \triangle BCD$.

(b) If $\triangle ACE \sim \triangle BCD$, prove $\overline{BD} \parallel \overline{AE}$.

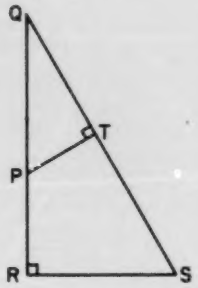
(c) If B is the midpoint of $\overline{AC}$, and D is the midpoint of $\overline{CE}$, prove $\triangle ACE \sim \triangle BCD$, and state the constant of proportionality.



6. Prove that two isosceles triangles are similar if a base angle of one is congruent to a base angle of the other.

7. Prove that two isosceles triangles are similar if the vertex angle of one is congruent to the vertex angle of the other.

8. If $\overline{PT} \perp \overline{QS}$, $\overline{QR} \perp \overline{RS}$, prove $\triangle QTP$ and $\triangle QRS$ are similar. Write a proportionality involving QP, PT, and TQ .

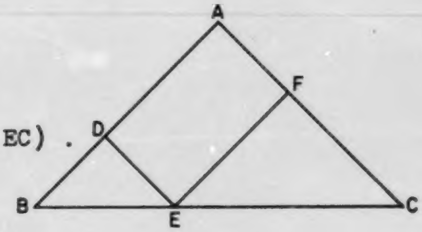


9. In $\triangle ABC$, $AB \cong AC$ and $\overline{ED} \perp \overline{AB}$, $\overline{EF} \perp \overline{AC}$. Prove

(a) $\triangle BDE \sim \triangle CFE$.

(b) $(BD, DE, EB) \underset{p}{=} (CF, FE, EC)$.

(c) $(DE, FE) \underset{p}{=} (EB, EC)$.

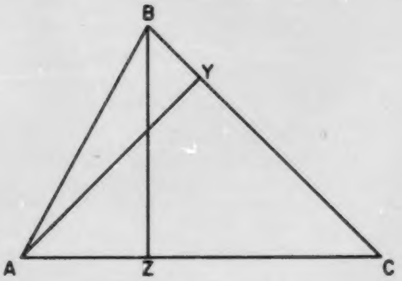


10. Hypothesis: In $\triangle ABC$,
$\overline{BZ} \perp \overline{AC}$,
$\overline{AY} \perp \overline{BC}$.

Prove:

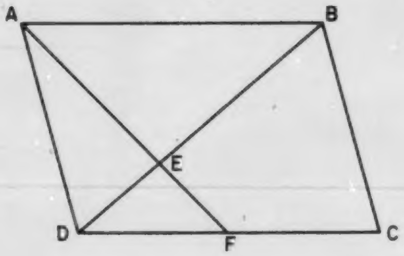
(a) $(BC, AC) \underset{p}{=} (BZ, AY)$.

(b) $(BC)(AY) = (AC)(BZ)$.



11. ABCD is a parallelogram. $DE = \frac{1}{2} EB$.

Prove $FD = \frac{1}{2} DC$.



12. If a tree casts a shadow of 13 feet and a six foot man nearby casts a shadow of two feet, what is the height of the tree?

13. The sides of a right triangle are 3, 4, and 5 units in length.

    (a) Prove that a triangle with sides 6, 8, and 10 units in length is also a right triangle.

    (b) Express algebraically, in terms of the proportionality constant k, the lengths of the sides of any other triangle similar to the given triangle, above.

14. If $ABCD$ is a parallelogram with E between A and B, F between C and D, and G the intersection of $\overline{EF}$ and $\overline{AC}$, prove that $(AG)(GF) = (CG)(GE)$.

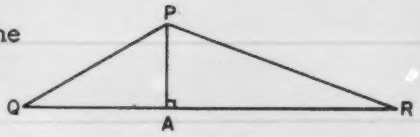
15. (a) If a ruler is held 28 inches from the eye, the diameter of the moon seems to be $\frac{1}{4}$ of an inch. If the distance to the moon is approximately 240,000 miles, give an approximation of the actual diameter of the moon.

    (b) When a ruler is held 28 inches from the eye, if the diameter of the sun also seems to be $\frac{1}{4}$ of an inch, what is the approximate diameter of the sun? The distance from the earth to the sun is about 93,000,000 miles.

7-6. Similarities in Right Triangles.

We pause to define two words which are useful in our study of right triangle similarities. They are "altitude" and "projection."

Let PQR be a triangle. By Theorem 5-11 there is a unique perpendicular to $\overleftrightarrow{QR}$ that contains P. Let A be the foot of the perpendicular from P to $\overleftrightarrow{QR}$. The segment $\overline{PA}$ is called an altitude of the triangle.



Thus in Figure a, $\overline{PA}$ is an altitude of $\triangle PQR$. We notice that the altitude is determined by the vertex P and by the side $\overline{QR}$. Another altitude of $\triangle PQR$, namely the altitude determined by Q, is the segment $\overline{QB}$. Name the altitude to the side $\overline{PQ}$.

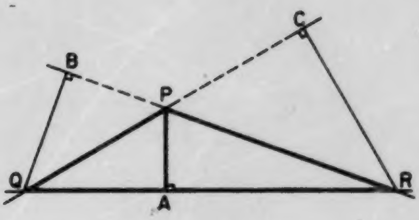
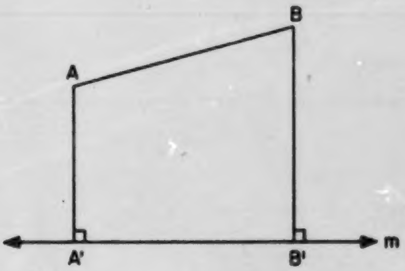


Figure a

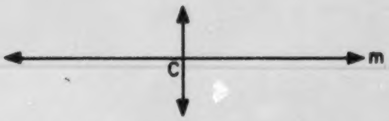
DEFINITION. The segment which joins a vertex of a triangle to the foot of the perpendicular from the vertex to the line containing the opposite side is called the altitude of the triangle from that vertex.

For emphasis we sometimes say that the segment is the altitude to the side.

Another notion related to perpendicularity is the idea of the projection of a segment on a line. Consider $\overline{AB}$ and line m, as shown. Let A' be the foot of the perpendicular from A to m. We sometimes say that A' is the projection of A on m. The point B', which is the foot of the perpendicular from B to m, is the projection of B on m. The projection of each point between A and B is some point between A' and B', and each point between A' and B' is the projection of some point between A and B.



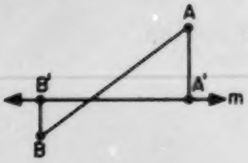
If a point C lies on a line m, then there is no perpendicular segment joining C to m. Nevertheless there is a line containing C and perpendicular to m. This line intersects m at C. So we consider the projection of C on m to be C itself.

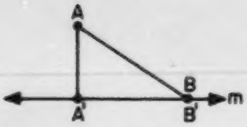


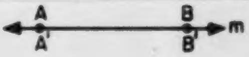
DEFINITIONS. The projection of a point on a line is the intersection of the given line and the line containing the given point and perpendicular to the given line.

The projection of a segment on a line is the set of all points which are projections on the given line of points in the given segment.

In each of the diagrams below, $\overline{A'B'}$ is the projection of $\overline{AB}$ on line m .





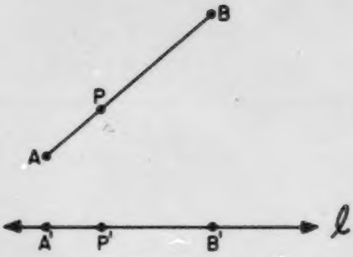


Since the altitude of a triangle from a vertex is the shortest segment from that vertex to the line containing the opposite side, the length of this altitude is the distance between the point and the line. In our later work, we often refer to the measure of an altitude of a triangle and for convenience, we simply use "altitude" when we mean "measure of the altitude." Similarly, we use "projection" to mean the "measure of a projection." However, if you remember that "altitude" and "projection" in one usage refer to a certain set of points and in the other to a certain number, the context should enable you to decide which meaning is intended.

Problem Set 7-6a

1. Draw three triangles; a right triangle, an obtuse triangle and an acute triangle. In each triangle, draw all three altitudes.

2. Given points A, P, B in that order and line ℓ . The projection of $\overline{AB}$ on ℓ is $\overline{A'B'}$ and the projection of $\overline{AP}$ on ℓ is $\overline{A'P'}$. If $AB = 12$, $A'B' = 9$, $AP = 4$, find $A'P'$ and $P'B'$.



3. Given: segment $\overline{PQ}$ whose projection on a line ℓ is $\overline{P'Q'}$.

   (a) Can the length of the projection of a segment be

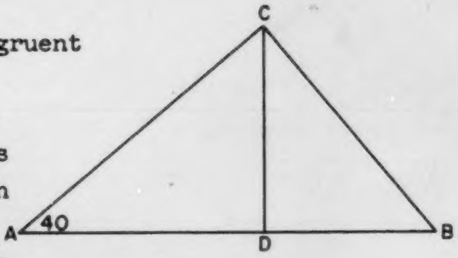
      (1) greater than the length of the segment?
      (2) equal to the length of the segment?
      (3) less than the length of the segment?

   (b) Is the projection of a segment always a segment?
   (c) If $P = P'$, $Q \neq Q'$, compare PQ with $P'Q'$.

4. In $\triangle ACB$, the altitude is drawn to the longest side, $\overline{AB}$, intersecting the interior of $\overline{AB}$ at D . Name the projection on $\overleftrightarrow{AB}$ of $\overline{AC}$; of $\overline{BC}$. What is the sum of the lengths of these projections?

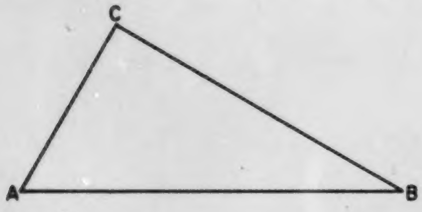
*5. Let ABC be a right triangle with right angle at C . Let $\overline{CD}$ be the altitude to the hypotenuse. Let $m\angle A = 40$.

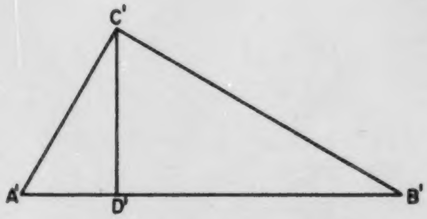
   (a) Find $m\angle ACD$, $m\angle DCB$, $m\angle CBD$.
   (b) Name two pairs of congruent angles in the figure.
   (c) Are these same angles congruent if $m\angle A$ is some number other than 40 ?
   (d) How many triangles are determined by A, B, C, D ?
   (e) Are any of these triangles similar?



Experiment

On a large sheet of paper draw two congruent right triangles, ABC and A'B'C', with right angles at C and C'.





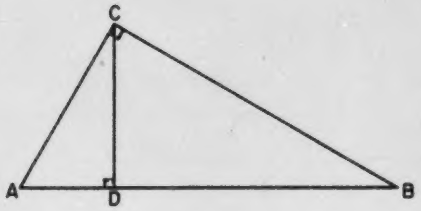
Draw the altitude C'D' of $\triangle A'B'C'$. Now, consider the three right triangles: $\triangle ABC$, $\triangle A'C'D'$, $\triangle C'B'D'$. Cut out these three triangles and match the right angles of the triangles. Do the triangles appear to be similar to each other? Try to match the remaining angles of the three triangles.

THEOREM 7-7. In any right triangle, the altitude to the hypotenuse separates the triangle into two triangles which are similar to each other and to the original triangle.

Proof: Let ABC be a right triangle with right angle at C, let $\overline{CD}$ be the altitude from C to $\overline{AB}$. We must prove that $\triangle ACD$, $\triangle CBD$, $\triangle ABC$ are all similar to one another. We do this by showing the correspondences

$$ACD \longleftrightarrow ABC \longleftrightarrow CBD$$

are similarities.



First of all, we do know by Theorem 6-20, that the altitude $\overline{CD}$ does intersect the hypotenuse, $\overline{AB}$, in its interior as pictured in the diagram, since $\overline{AB}$ is the longest side of the triangle.

Now, we prove the triangles are similar by considering in order each of the following pairs:

(1) $\triangle ACD$ and $\triangle ABC$.
(2) $\triangle ABC$ and $\triangle CBD$.
(3) $\triangle ACD$ and $\triangle CBD$.

First, $\angle DAC \cong \angle CAB$ by the reflexive property. Also $\angle CDA \cong \angle BCA$ since they are both right angles. Thus $\triangle ACD \sim \triangle ABC$ by the A.A. Similarity Theorem. In like manner, $\triangle ABC \sim \triangle CBD$.

Then, using the transitive property of similarity, we know that $\triangle ACD \sim \triangle CBD$.

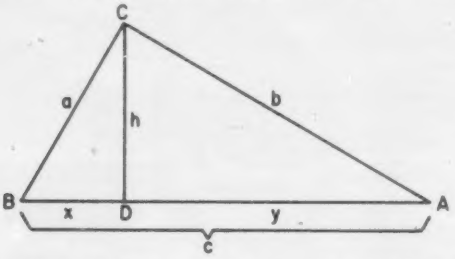
We conclude that all three triangles are similar to one another in the correspondence given; that is, $\triangle ACD \sim \triangle ABC \sim \triangle CBD$. Notice that we acquire more information from this correspondence of the vertices than is provided by the statement of the theorem. This information leads to the following interesting relationship among the lengths of various segments of a right triangle. You will be asked to prove these corollaries in the problem set.

Corollary 7-7-1. The square of the altitude to the hypotenuse of a right triangle is equal to the product of the projections of the legs on the hypotenuse.

Corollary 7-7-2. The square of the length of either leg of a right triangle is equal to the product of the lengths of the hypotenuse and the projection of that leg on the hypotenuse.

Problem Set 7-6b

1. Given $\triangle ABC$ with hypotenuse $\overline{AB}$ and altitude $\overline{CD}$. In Theorem 7-7, we proved $\triangle ACB \sim \triangle CDB \sim \triangle ADC$.



Complete the chart below to show the lengths of the sides of $\triangle CDB$ and $\triangle ADC$ that correspond to the lengths c, a, b of the sides of $\triangle ACB$:

$\triangle ACB$	$\triangle CDB$	$\triangle ADC$
c		
a		
b		

2. Refer to the diagram in Problem 1 and write Corollaries 7-7-1 and 7-7-2 as formulas involving $a, b, c, h, x,$ and y.

3. Prove: (a) Corollary 7-7-1.
   (b) Corollary 7-7-2.

4. Refer to Problem 1. Prove that $h = \frac{ab}{c}$.

5. Referring to the diagram in Problem 1, find each of the following measures.

   (a) If $x = 2$, $y = 8$, find h.
   (b) If $x = 1$, $c = 10$, find h.
   (c) If $x = 7$, $y = 7$, find h.
   (d) If $x = .5$, $h = 2$, find c.
   (e) If $x = \sqrt{2}$, $y = \sqrt{8}$, find h.
   (f) If $x = 4$, $c = 9$, find a.
   (g) If $x = 3$, $y = 9$, find a.
   (h) If $b = 4$, $x = 6$, find y.
   (i) If $b = 8\sqrt{5}$, $y = 16$, find x.
   (j) If $c = 25$, $h = 10$, find x.

7-7. The Pythagorean Theorem.

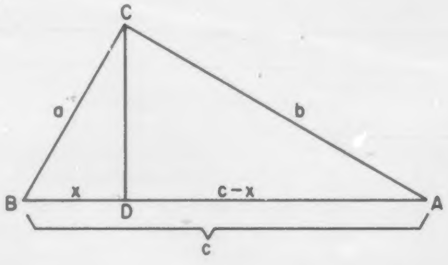
The theorem which bears the name of the ancient Greek scholar Pythagoras is probably the most famous of all mathematical theorems. The Egyptians learned experimentally that if three ropes have lengths measuring $3, 4, 5$, respectively, then the ends can be joined to form a right triangle. This knowledge is very useful in making corners square in construction and surveying operations. Other triplets of lengths may also be used, as for instance, $5, 12, 13$. You may notice that $3^2 + 4^2 = 5^2$ and that $5^2 + 12^2 = 13^2$. The Pythagorean Theorem tells us the converse of the principle used by the Egyptians, namely, that if a and b are measures of the legs of a right triangle, and c the measure of the hypotenuse, then $a^2 + b^2 = c^2$. At this time, you should reread the discussion of the Pythagorean Theorem, its converse, and related material in Sections 1-2, 1-3, 1-7.

The Pythagorean Theorem is not only a key principle in construction and surveying, but it has been found extremely important in many scientific and mathematical studies. For this reason, it has attracted the attention of numerous mathematicians, and more proofs have been found for this theorem than for any other.

THEOREM 7-8. (The Pythagorean Theorem) In any right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the two legs.

Proof: Given: $\triangle ABC$ with a right angle at C. Measures a, b, c of the sides respectively opposite vertices A, B, C.

To prove: $a^2 + b^2 = c^2$.

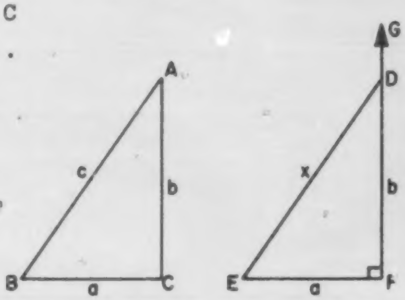


Statements	Reasons
1. The foot D of the altitude from C is an interior point of $\overline{BA}$.	1. Theorem 6-20.
2. Let $BD = x$ and $DA = c - x$.	2. The Betweenness-Distance Theorem.
3. $a^2 = cx$.	3. Corollary 7-7-2.
4. $b^2 = c^2 - cx$.	4. Corollary 7-7-2.
5. $a^2 + b^2 = c^2$.	5. Addition property of equality.

The converse of the Pythagorean Theorem is also valid. We now prove it.

THEOREM 7-9. (Converse of Pythagorean Theorem) If the square of the length of one side of a triangle is equal to the sum of the squares of the lengths of the other two sides, then the triangle is a right triangle with the right angle opposite the first side.

Proof: In the triangle ABC , let the measures of the sides opposite the vertices A, B, C be the respective numbers a, b, c . It is given that $a^2 + b^2 = c^2$; we are to prove that $\angle C$ is a right angle.



There is a right triangle DEF such that $EF = a$, $m\angle F = 90$, and $DF = b$, because:

1. there are points E, F such that $EF = a$, by the Ruler Postulate;
2. there is a right angle $\angle EFG$, by the Protractor Postulate; and
3. on $\overrightarrow{FG}$ there is a point D such that $DF = b$, by the Point Plotting Theorem.

If ED = x , then the Pythagorean Theorem, applied to $\triangle DEF$, tells us that $a^2 + b^2 = x^2$. Comparing this with the hypothesis that $a^2 + b^2 = c^2$, we conclude that $c = x$, since both c and x are positive. Thus, $\triangle ABC \cong \triangle DEF$ (why?) and $\angle C$ is a right angle (why?).

In everyday speech, we often shorten the statements of the Pythagorean Theorem and its converse by saying "the square of a side" rather than "the square of the length of a side." Using this phraseology, we may summarize the two theorems of this section in the following "if and only if" version.

> A triangle is a right triangle if and only if the square of one side is the sum of the squares of the other two sides.

Problem Set 7-7

1. In each of the following, suppose that the given three numbers are the measures of the sides of a triangle. Determine, in each case, whether the triangle is a right triangle, and explain why.

   (a) 3, 4, 5 .

   (b) 4, $4\sqrt{3}$, 8 .

   (c) x, x, $x\sqrt{3}$.

   (d) 6, 8, 10 .

   (e) 12, 13, 5 .

   (f) 20, 16, 11 .

   (g) $\frac{5\sqrt{2}}{2}$, 5, 5 .

   (h) $1\frac{1}{2}$, 2, $2\frac{1}{2}$.

   (i) 24, 12, 26 .

   (j) 7, 24, 25 .

   (k) 30, 40, 56 .

   (l) $6\sqrt{3}$, $12\sqrt{3}$, 18 .

   (m) 9ab, 12ab, 15ab .

   (n) 8, 15, 17 .

   (o) 29, 20, 21 .

2. In the following, c is the length of the hypotenuse and a and b are lengths of the legs of a right triangle. Find:

   (a) a if $c = 20$ and $b = 12$.
   (b) c if $a = 9$ and $b = 7$.
   (c) a if $c = 15$ and $b = 5$.
   (d) b if $a = 10$ and $c = 16$.
   (e) c if $a = 1.5$ and $b = 2$.

3. A ladder 18 feet long reaches to a window sill of the side of a house. If the window sill is 15 feet above the ground, how far from the side of the house is the foot of the ladder?

4. Two cars leave a certain point at the same time. One travels north at the rate of 30 miles an hour; the other travels west at the rate of 40 miles an hour. How far apart are the cars at the end of $1\frac{1}{2}$ hours?

5. Find the length of the altitude to the base of an isosceles triangle whose sides measure 25, 25, 48 .

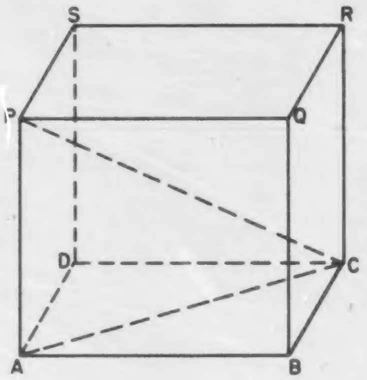
6. Find the length of the congruent sides of an isosceles triangle if the length of the base of the triangle is 12 inches and the altitude to the base is 16 inches.

7. Find the length of a side of an equilateral triangle if the altitude to that side is 4 .

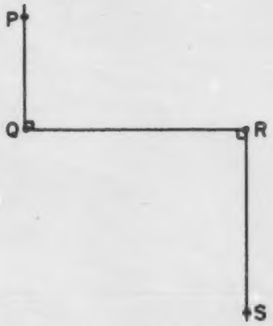
8. Find the length of the diagonal of the floor of a classroom if the floor is 16 feet long and 12 feet wide.

9. The figure to the right is a rectangular solid with $\overline{AB} \perp \overline{BC}$ and $\overline{AP} \perp \overline{AC}$.

   Find PC if $AB = 8$, $BC = 4$, and $AP = 8$.



10. In the diagram of a plane figure, the angles at Q and R are right angles; PQ = 3, QR = 6, and RS = 5. Find PS.



11. Two vertical poles on level ground are 12 feet apart and their tops are 10 and 15 feet, respectively, above the ground. Find the distance between the tops of the poles.

*12. Find the length of the hypotenuse of a right triangle each of whose legs has a measure of 1 .

*13. The hypotenuse of an isosceles right triangle has a measure of $6\sqrt{2}$. Find the measure of each leg.

*14. If the length of one leg of a right triangle is half the length of the hypotenuse, and the hypotenuse is 2x , find the lengths of the two legs in terms of x .

15. Let u, v, w be positive numbers such that $u^2 + v^2 = w^2$. Prove that there is a right triangle whose sides have respective lengths u, v, w .

16. Show that a triangle whose sides measure $x^2 - 1$, 2x , and $x^2 + 1$, respectively, (where $x > 1$) is a right triangle. What are these three numbers if $x = 2$? $x = 4$? $x = 6$?

17. Show that the numbers $u^2 + v^2$, 2uv , and $u^2 - v^2$, (where $0 < v < u$) are the measures of the three sides of a right triangle. What are these numbers if $u = 5$ and $v = 2$?

18. If the measures of the three sides of a right triangle are all integers, the three integers are sometimes called a "Pythagorean Triple." Find at least one "Pythagorean Triple" that is not listed in Problem 1 above.

*19. If two legs of one right triangle are proportional to the two legs of any other right triangle, are the triangles similar? Why?

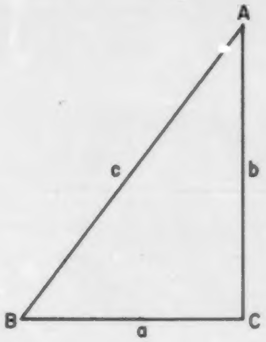
*20. If a leg and the hypotenuse of one right triangle are proportional to either leg and the hypotenuse of a second right triangle, are the triangles similar? Explain.

7-8. Special Right Triangles.

Certain right triangles appear frequently in problems of the physical world such as in engineering. It is worth making their acquaintance so that you may recognize them and be able to use them in developing some computational short cuts. These triangles may be classified into sets of triangles, each set containing only triangles that are similar to one another. We shall list some of these sets.

I. The 3, 4, 5 right triangles.

Since $3^2 + 4^2 = 5^2$, the converse of the Pythagorean Theorem tells us that a triangle whose sides have measures 3, 4, 5 is a right triangle. Any triangle similar to a right triangle is itself a right triangle; why? Thus, if any positive number k is used as a proportionality constant, we conclude that a triangle whose sides measure $3k, 4k, 5k$ is a right triangle. For instance, any triangle whose sides measure 6, 8, 10 or 150, 200, 250, is a member of this set of right triangles.



Example 1.

Given: In right triangle ABC with right angle at C , $a = 18$, $b = 24$. To find c , we note that $a = 3 \cdot 6$, $b = 4 \cdot 6$. Therefore $c = 5 \cdot 6$ or 30 .

Example 2.

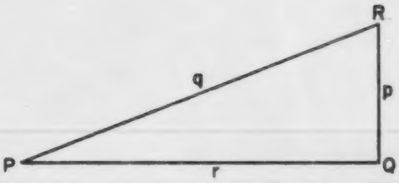
In right triangle ABC , $c = 75$ and $a = 45$. To find b we note that $c = 5 \cdot 15$, $a = 3 \cdot 15$. Therefore $b = 4 \cdot 15$ or 60 .

II. The 5, 12, 13 right triangles.

Explain why a triangle whose sides have measures 5, 12, 13 is a right triangle. Explain why a triangle whose sides have lengths 5k, 12k, 13k, for some positive number k, is a right triangle. Explain why all these triangles are similar to one another. Give other examples of three numbers which are the respective measures of the sides of triangles in this set of similar right triangles.

Example 1.

In $\triangle PQR$, $\angle Q$ is a right angle, $PR = q$, $RQ = p$, $PQ = r$. Suppose $p = 15$ and $r = 36$. To find q, we note that $p = 5 \cdot 3$, $r = 12 \cdot 3$; therefore, $q = 13 \cdot 3$ or 39.



Example 2.

Suppose $q = 78$ and $p = 72$. To find r, we note that $q = 13 \cdot 6$, $p = 12 \cdot 6$ and therefore $r = 5 \cdot 6$ or 30.

III. The 30, 60, 90 triangles.

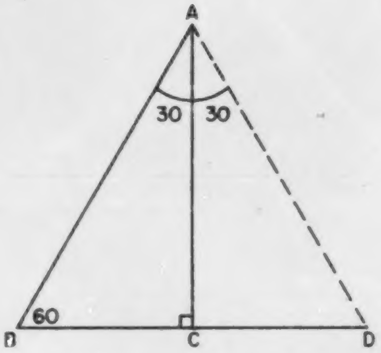
Unlike the sets of triangles in (I) or (II), these triangles have angles rather than sides whose measures are "convenient" numbers. Suppose in $\triangle ABC$, $m\angle C = 90$, $m\angle B = 60$, and $m\angle A = 30$. Let D be a point such that B and D are on opposite sides of $\overleftrightarrow{AC}$ and $m\angle CAD = 30$, and let D be the point of intersection of $\overrightarrow{AD}$ and $\overleftrightarrow{BC}$.

We can prove

$$\triangle ABC \cong \triangle ADC. \quad \text{Why?}$$

Therefore $m\angle D = m\angle B = 60$.

Also $m\angle BAD = 60$.

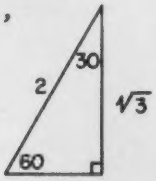


Hence, $\triangle BAD$ is equiangular and therefore equilateral. If $BC = a$, then $BD = BA = 2a$, and $AC^2 + a^2 = (2a)^2$.

This equation permits the conclusion $AC = a\sqrt{3}$. Therefore, we see that $(BC, CA, AB) \underset{p}{=} (1, \sqrt{3}, 2)$.

Conversely, given a triangle $A'B'C'$, if $(B'C', C'A', A'B') \underset{p}{=} (1, \sqrt{3}, 2)$ then $\triangle A'B'C'$ is similar to right triangle ABC above by the S.S.S. Similarity Theorem. Thus, $\triangle A'B'C'$ is also a right triangle. In fact, since all corresponding angles must be congruent, $m\angle A' = 30$, $m\angle B' = 60$, $m\angle C' = 90$.

We summarize this discussion in a theorem.

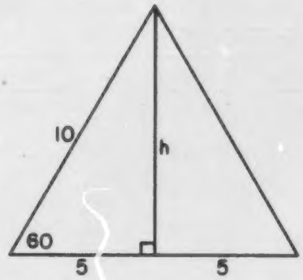


__THEOREM__ __7-10__. The triangle ABC is a right triangle with $m\angle A = 30$, $m\angle B = 60$, and $m\angle C = 90$ if and only if $(BC, CA, AB) \underset{p}{=} (1, \sqrt{3}, 2)$.

Because of Theorem 6-18, we conclude $BC < CA < AB$. We can therefore in the statement of this theorem use a proportionality, in which we refer to lengths of sides: (shorter leg, longer leg, hypotenuse) $\underset{p}{=} (1, \sqrt{3}, 2)$.

__Example__ __1__.

Find the length of an altitude of an equilateral triangle if the length of one of its sides is 10 . $(5, h, 10) \underset{p}{=} (1, \sqrt{3}, 2)$. The constant of proportionality is 5 . Therefore $h = 5\sqrt{3}$.



__Example__ __2__.

Find the measures of the sides of a 30, 60, 90 triangle if the length of the longer leg is 8 . Let a = length of the shorter leg and c = length of the hypotenuse, then

$$(a, 8, c) \underset{p}{=} (1, \sqrt{3}, 2) .$$

The constant of proportionality is $\frac{8}{\sqrt{3}}$; therefore, $a = \frac{8}{\sqrt{3}}$ and $c = 2 \cdot \frac{8}{\sqrt{3}}$.

IV. The $1, 1, \sqrt{2}$ right triangles (sometimes called the isosceles right triangles or the 45, 45, 90 triangles).

THEOREM 7-11. The triangle ABC is a right triangle with right angle at C, and with AC = BC, if and only if $(AC, BC, AB) \underset{p}{=} (1, 1, \sqrt{2})$.

The proof of this theorem is left as a problem.

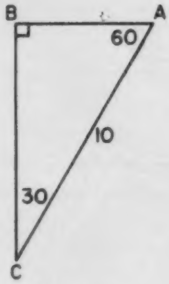
Example.

Find the length of a side of a square if the length of its diagonal is 10. $(a, a, 10) \underset{p}{=} (1, 1, \sqrt{2})$. The constant of proportionality is $\frac{10}{\sqrt{2}}$ or $5\sqrt{2}$. Therefore, $a = 5\sqrt{2}$.

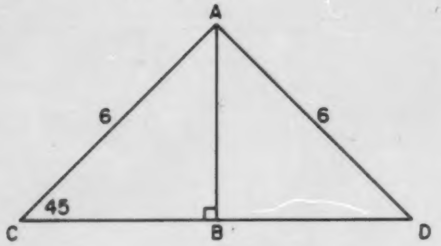
Problem Set 7-8

1. Find AB and BC in each of the following figures.

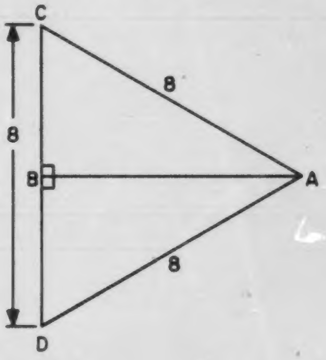
(a)



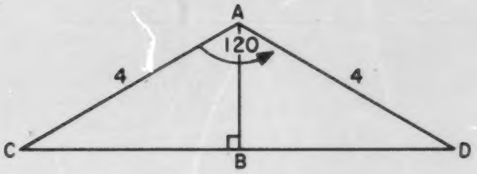
(b)



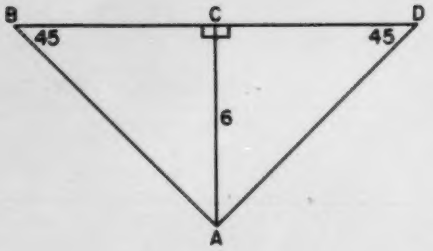
(c)



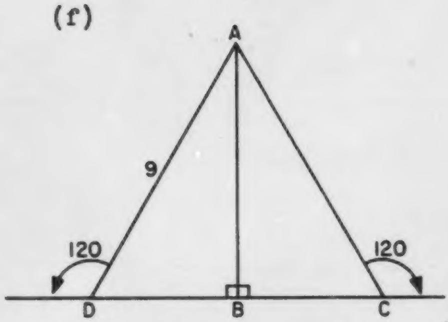
(d)



(e)

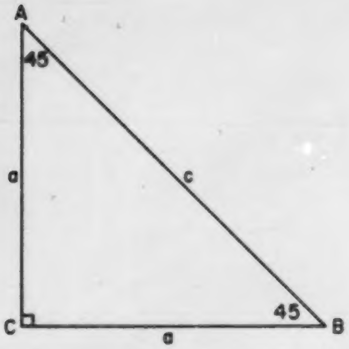


(f)



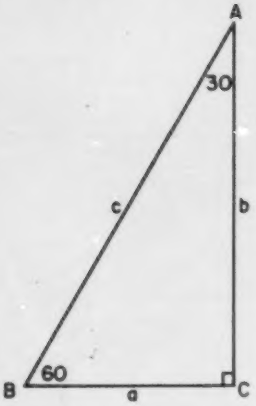
2. Prove Theorem 7-11.

3. Complete the table for the right isosceles triangle in the diagram.



45-45-90 triangle		
	a	c
(a)	10	___
(b)	5	___
(c)	___	9
(d)	___	6
(e)	___	$3\sqrt{2}$
(f)	$5\sqrt{2}$	___

4. Complete the table for the 30-60-90 triangle in the diagram.



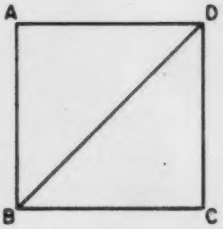
30-60-90 triangle			
	a	b	c
(a)	10	___	___
(b)	5	___	___
(c)	___	9	___
(d)	___	$9\sqrt{3}$	___
(e)	___	___	12
(f)	___	___	$12\sqrt{3}$

5. Find the length of the altitude of an equilateral triangle if the length of each side is 6 .

6. Find the length of a side of an equilateral triangle if the length of the altitude is 6 .

7. In the figure, ABCD is a square with AB = BC = CD = DA . $\angle A$, $\angle B$, $\angle C$, $\angle D$ are right angles. $\overrightarrow{BD}$ bisects $\angle B$.

   Find BD if AB is 6 .



8. In Problem 7, find the length of a side of the square if the length of the diagonal is 6 .

9. A baseball diamond is a square whose sides are 90 feet long. What is the distance a ball travels when thrown from first to third base?

10. A boy is flying a kite with a string 300 feet long, which is attached to a stake on the ground. If all the string is out and the string makes an angle of 30 degrees with the ground, how high is the kite?

11. In each of the following are given the lengths of a leg and the hypotenuse of a right triangle. Which of the measures belong to a triangle similar to the 3, 4, 5 triangle? to the 5, 12, 13 triangle? to the $1, \sqrt{3}, 2$ triangle? to the $1, 1, \sqrt{2}$ triangle?

    (a) 6, 10
    (b) 12, 15
    (c) 24, 25
    (d) 15, 39
    (e) 3, 6
    (f) $3, 2\sqrt{3}$
    (g) 8, 10
    (h) 1.5, 2.5
    (i) 6, 6.5
    (j) 24, 26
    (k) $3\sqrt{2}, 5\sqrt{2}$
    (l) $\sqrt{2}, 2$

12. In each part of Problem 11, find the length of the side which is not given.

7-9. Summary.

In Chapter 5, the idea of "same size and shape" of physical objects suggested the mathematical concept of a congruence. Congruence was developed as a one-to-one correspondence between the vertices of two triangles and relied on properties of equality.

In this chapter, the idea of "same shape" suggested the mathematical concept of a similarity. Similarity was developed as a one-to-one correspondence between vertices of polygons and relied on properties of equality and properties of proportionality. We learned, too, that a congruence is a similarity.

We proved three triangle similarity theorems comparable to the S.S.S., S.A.S., and A.S.A. congruence postulates. The culmination of this development was the Pythagorean Theorem.

Some of the new terms in this chapter are:

proportionality
proportion
constant of proportionality
projection
similarity
altitude

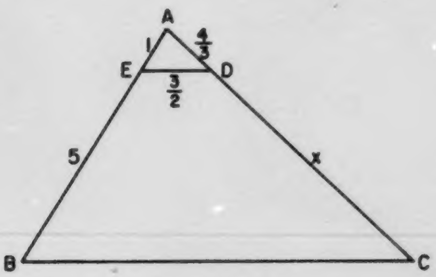
Review Problems

1. When we say "a, b are proportional to c, d, with proportionality factor k", we are saying in effect: "_____ = _____ and _____ = _____ ."
2. Given that 3, x are proportional to 39, 65:
   (a) Find x .
   (b) What is the proportionality constant in the above proportionality relation?
   (c) Complete: 3, x, 3 + x are proportional to 39, _____, _____ .
3. The number _____ is not permitted to be a constant of proportionality. Why?
4. If the three angles of one triangle have measures proportional to the measures of corresponding angles of another triangle, what is the proportionality constant? Explain.

5. If the lengths of the three sides of one triangle are proportional to the lengths of the corresponding sides of a second triangle, the proportionality constant may be any number belonging to the set of all _____ numbers. (Fill the blank with one word.)

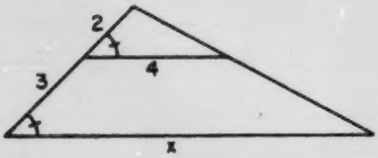
6. Under what circumstances is a correspondence between the vertices of triangles said to be a similarity?

7. In the triangle ABC pictured at the right, $\overline{ED} \parallel \overline{BC}$, and lengths of segments are as indicated. Find x. Find AC. Find BC.

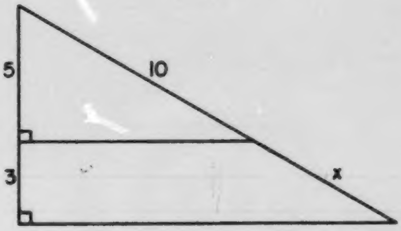


8. Find the value of x in each of the following problems.

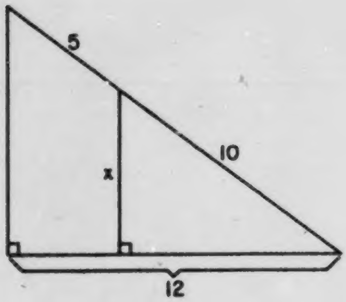
(a)



(c)



(b)



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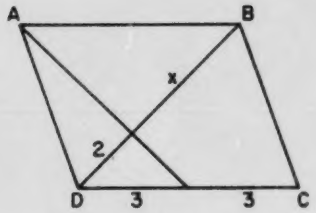
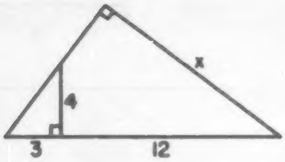
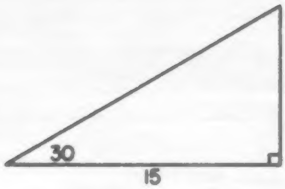


ABCD is a parallelogram.

(e)



(i)



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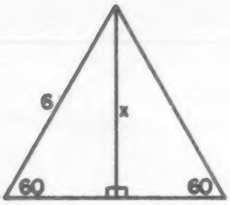
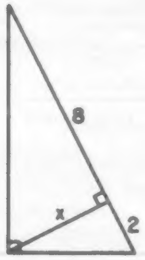
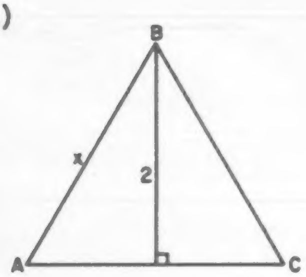


(j)

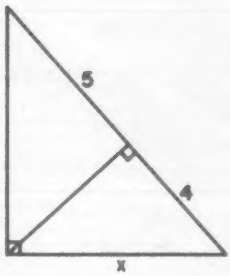


(g)

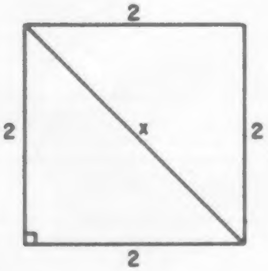


△ABC is equilateral.

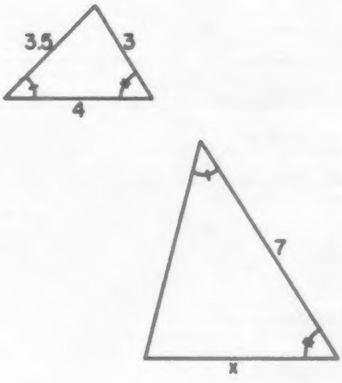
(k)



(h)

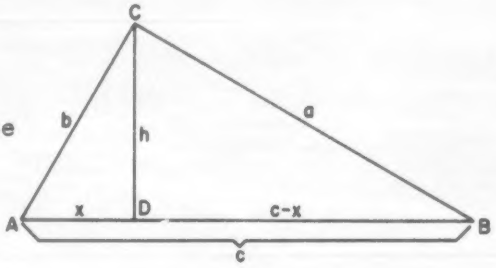


(l)



9. $\triangle ABC \sim \triangle RST$ with proportionality constant k . $\triangle XYZ \sim \triangle RST$ with proportionality constant m . Is $\triangle ABC \sim \triangle XYZ$? If so, what is the proportionality constant?

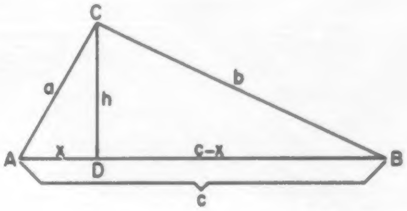
10. In this figure $\overline{CD}$ is the altitude to the hypotenuse of right triangle ABC . Complete these proportionality relations:



$(b, h, x) \underset{p}{=} (a, \_\_, \_\_)$

$(a, \_\_, \_\_) \underset{p}{=} (c, \_\_, \_\_)$.

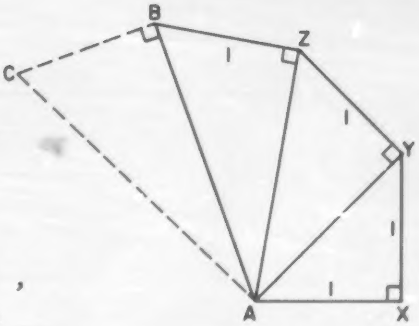
11. In this figure $\triangle ADC \sim \triangle CDB$ with proportionality factor k . Thus $x = kh$. Express a in terms of k and the length of some segment. Do the same for h .



12. In $\triangle ABC$, $AB = 2$, $AC = 1$, $BC = \sqrt{3}$. Find $m\angle A$, $m\angle B$, $m\angle C$.

13. In $\triangle ABC$, $AB = 8$, $AC = CB = 4\sqrt{2}$. Find $m\angle A$, $m\angle B$, $m\angle C$.

14. Consider the diagram with right angles and lengths as marked.



(a) Find AY, AZ, and AB .

(b) If you continue the pattern established in this figure making $m\angle ABC = 90$ and $BC = 1$, what would be the length of $\overline{AC}$?

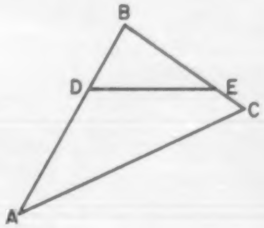
(c) If the distance between two dots on the blackboard (or sheet of paper) is used as one unit of measure, locate two dots such that the distance between them is $\sqrt{6}$.

15. A rectangle is a parallelogram whose consecutive sides are perpendicular. Find the measure of the diagonal of a rectangle if two sides are 5 and 12 .

16. In $\triangle ABC$, $BD = 2$, $BC = 4$, $BE = 3$, $BA = 6$, $DE = 3.5$.

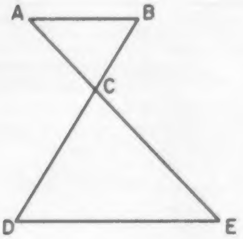
(a) Find AC .

(b) Is $\overline{DE} \parallel \overline{AC}$? Explain.



17. The sides of a triangle measure 5, 6, and 8 . Find the measures of the corresponding sides of a similar triangle whose perimeter is 57 .

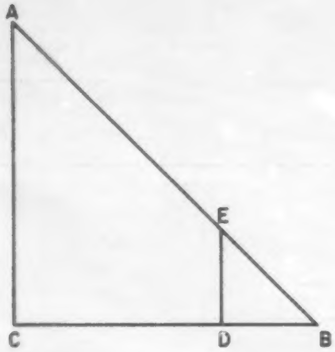
18.



Hypotheses: $\overline{AE}$ intersects $\overline{BD}$ at C .

$$\frac{AC}{CE} = \frac{BC}{CD} .$$

Prove: $\overline{AB} \parallel \overline{DE}$.

19. 

Hypothesis: $\overline{ED} \perp \overline{BC}$, $\overline{AC} \perp \overline{CB}$.

Prove: $\triangle EBD \sim \triangle ABC$.

20. The hypotenuse of a right triangle measures 13.6 and a leg has a length 6.4 . Find the length of the other leg.

Appendix I

A CONVENIENT SHORTHAND

There was a time when algebra was all written out in words. In words, you might state an algebraic problem in the following way:

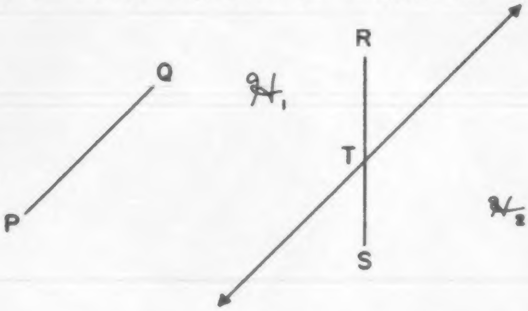
"If you square a certain number, add five times the number, and then subtract six, the result is zero. What are the possibilities for this number?"

This problem can be more briefly stated in the following form:

"Find the roots of the equation $x^2 + 5x - 6 = 0$."

The notation of algebra is a very convenient shorthand. A similar shorthand has been invented for talking about sets. It saves a lot of time and space, once you get used to it, and it is all right to use it in your written work, unless your teacher objects.

Let us start with a picture, and say various things about it first in words and then in shorthand.



Here we see a line ℓ , which separates the plane $\mathcal{E}$ into two halfplanes $\mathcal{H}_1$ and $\mathcal{H}_2$. Now let us say some things in two ways.

In Words	In Shorthand
1. The segment $\overline{PQ}$ lies in $\mathcal{H}_1$.	1. $\overline{PQ} \subset \mathcal{H}_1$.
2. The intersection of $\overline{RS}$ and ℓ is T .	2. $\overline{RS} \cap \ell = T$.

The shorthand expression $\overline{PQ} \subset \mathcal{H}_1$ is pronounced in exactly the same way as the expression on the left of it. In general, when we write $A \subset B$, this means that the set A lies in the set B .

An expression of the type $A \cap B$ denotes the intersection of the sets A and B . The symbol "$\cap$" is pronounced "cap," because it looks a little like a cap. Notice that the sets $\overline{PQ}$ and $\overline{RS}$ do not intersect. If we agree to write $\emptyset$ for the empty set, then we can express this fact by writing

$$\overline{PQ} \cap \overline{RS} = \emptyset .$$

Similarly,

$$\overline{PQ} \cap \ell = \emptyset$$

and

$$\overline{PQ} \cap \mathcal{H}_2 = \emptyset .$$

Of course, $\overline{PQ}$ is a set which lies in $\mathcal{H}_1$. But the point P above is a member of $\mathcal{H}_1$. We write this in shorthand like this

$$P \in \mathcal{H}_1 .$$

This is pronounced "P belongs to $\mathcal{H}_1$."

The union of two sets A and B is written as $A \cup B$. This is pronounced "A cup B ." In the same way, we write $A \cup B \cup C$ for the union of three sets. For example, in the figure on the previous page, the plane $\mathcal{E}$ is the union of $\mathcal{H}_1$, $\mathcal{H}_2$, and ℓ . We can therefore write

$$\mathcal{E} = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \ell .$$

Problem Set I

Consider the sets, A, B, C, and so on, defined in the following way:

A is the set of all doctors.
B is the set of all lawyers.
C is the set of all tall people.
D is the set of all people who can play the violin.
E is the set of all people who make a lot of money.
F is the set of all basketball players.

Write shorthand expressions for the following statements:

1. All basketball players are tall.
2. No doctor is a lawyer.
3. No violinist makes a lot of money, unless he is tall.
4. No basketball player is a violinist.
5. Everyone who is both a doctor and a lawyer can also play the violin.
6. Every basketball player who can play the violin makes a lot of money.
7. The man X is a tall violinist.
8. The man Y is a prosperous lawyer.
9. The man Z is a tall basketball player.

Appendix II

POSTULATES FOR ADDITION AND MULTIPLICATION

The mathematics of combining real numbers by means of addition, subtraction, multiplication, and division, has its basis in the following definitions and postulates. Several theorems are deduced from the postulates at the end of the Appendix.

POSTULATES FOR EQUALITY

The Substitution Property of Equality

One name for an object may be substituted for another name for that object in any statement about that object without changing the truth value of the statement.

Reflexive Property of Equality

For all x , $x = x$.

Symmetric Property of Equality

For all x and y , if $x = y$ then $y = x$.

Transitive Property of Equality

For all $x, y, z,$ if $x = y$ and $y = z$ then $x = z$.

POSTULATES FOR ADDITION AND MULTIPLICATION OF REAL NUMBERS

Closure under Addition

For all x and y , $x + y$ is a unique real number.

Associative Law for Addition

For all $x, y, z,$ $x + (y + z) = (x + y) + z$.

Commutative Law for Addition

For all x and y, $x + y = y + x$.

Additive Identity

There is a unique number 0 such that for all x, $x + 0 = x$.

Additive Inverse

For each x there is a unique number $-x$ such that $x + (-x) = 0$.

Closure under Multiplication

For all x and y, xy is a unique real number.

Associative Law for Multiplication

For all x, y, z, $x(yz) = (xy)z$.

Commutative Law for Multiplication

For all x and y, $xy = yx$.

Multiplicative Identity

There is a unique number 1 such that for all x, $x \cdot 1 = x$.

Multiplicative Inverse

For each x except 0, there is a unique number $\frac{1}{x}$, such that $x \cdot \frac{1}{x} = 1$.

Distributive Law

For all x, y, z, $x(y + z) = xy + xz$.

DEFINITIONS

Definition of Subtraction

For all x and y, $x - y = x + (-y)$.

Definition of Division

For all x, and for all y except 0, $\frac{x}{y} = x \cdot \frac{1}{y}$.

THEOREMS

THEOREM II-1. For all x and y, if $y = -x$, then $-y = x$.

Proof:

$y = -x$	hypothesis
$x + (-x) = 0$	additive inverse
$x + y = 0$	substitution
$y + x = 0$	commutative law for addition
$x = -y$	additive inverse
$-y = x$	symmetric property of equality

THEOREM II-2. For all x, $x \cdot 0 = 0$.

Proof:

$x = x \cdot 1$	multiplicative identity
$= x(1 + 0)$	additive identity
$= x \cdot 1 + x \cdot 0$	distributive law
$= x + x \cdot 0$	multiplicative identity
$x = x + x \cdot 0$	trans. property of equality
$x \cdot 0 = 0$	additive identity

THEOREM II-3. For all x and y, $x(-y) = -xy$.

Proof:

$xy + x(-y) = x[y + (-y)]$	distributive law
$= x \cdot 0$	additive inverse
$= 0$	Theorem II-2
$xy + x(-y) = 0$	trans. property of equality
$x(-y) = -xy$	additive inverse

THEOREM II-4. For all x, $x(-1) = -x$.

Proof:

$x(-1) = -1 \cdot x$	Theorem II-3
$= -x$	multiplicative identity

THEOREM II-5. For all x,y,z, if $x + y = z$ then $x = z - y$.

Proof:

$(x + y) + (-y) = x + [y + (-y)]$	associative law for addition
$= x + 0$	additive inverse
$= x$	additive identity
$(x + y) + (-y) = x$	trans. property of equality
$x + y = z$	hypothesis
$z + (-y) = x$	substitution
$z - y = x$	definition of subtraction

THEOREM II-6. For all x and y, if $xy = 0$, then $x = 0$ or $y = 0$.

Proof: It is enough to show that if $x \neq 0$, then $y = 0$. Suppose, then, that $x \neq 0$.

$xy = 0$	hypothesis
$\frac{1}{x}(xy) = \frac{1}{x} \cdot 0$	closure under multiplication
$\frac{1}{x}(xy) = 0$	Theorem II-2
$(\frac{1}{x} \cdot x)y = 0$	assoc. law for multiplication
$1 \cdot y = 0$	multiplicative inverse
$y = 0$	multiplicative identity

THEOREM II-7. For all x, y, z, if $xy = xz$ and $x \neq 0$, then $y = z$.

Proof:

$xy = xz$	hypothesis
$xy + (-xz) = xz + (-xz)$	closure for addition
$xy + (-xz) = 0$	additive inverse
$xy + x(-z) = 0$	Theorem II-3
$x[y + (-z)] = 0$	distributive law
$x \neq 0$	hypothesis
$y + (-z) = 0$	Theorem II-6
$y - z = 0$	definition of subtraction

THEOREM II-8. For all x and y, $y + (x - y) = x$.

Proof:

$y + (x - y) = (x - y) + y$	commutative law for addition
$= [x + (-y)] + y$	definition of subtraction
$= x + [(-y) + y]$	association law for addition
$= x + [y + (-y)]$	commutative law for addition
$= x + 0$	additive inverse
$= x$	additive identity

Problem Set II

1. Prove each of the following statements.
   - (a) For all x and y, $(-x)(-y) = xy$.
   - (b) For all x, y, z, $x(y - z) = xy - xz$.
   - (c) For all x, y, z, if $x - y = z$ then $x = y + z$.
   - (d) For all x, y, u, v, it is true that $(x + y)(u + v) = (xu + xv) + (yu + yv)$.
2. Given the definitions:

   $2 = 1 + 1$
   For all x, $x^2 = x \cdot x$
   For all x, y, z, $x + y + z = (x + y) + z$
   For all x, y, z, $xyz = (xy)z$.

   Prove:
   For all x and y, $(x + y)^2 = x^2 + 2xy + y^2$.

3. Prove: For all x and y, $(x + y)(x - y) = x^2 - y^2$.

4. Given the definitions: For all y except 0 , $y^{-1} = \frac{1}{y}$. Prove each of the following statements.

   (a) For all x except 0 , and for all y except 0 , $(xy)^{-1} = x^{-1} \cdot y^{-1}$.

   (b) For all x and z , for all y except 0 , and for all w except 0 , $\frac{x}{y} \cdot \frac{z}{w} = \frac{xz}{yw}$.

   (c) For all x, y, z such that $y \neq 0$, $z \neq 0$, $\frac{x}{y} = \frac{xz}{yz}$.

   (d) For all x except 0 , $(-x)^{-1} = -x^{-1}$.

   (e) For all x and y and for all z except 0 , $\frac{x}{z} + \frac{y}{z} = \frac{x + y}{z}$.

   (f) For all x and for all y except 0 , $\frac{x}{-y} = \frac{-x}{y} = -\frac{x}{y}$.

Appendix III

THEOREMS ON INEQUALITIES

In addition to the postulates for addition and multiplication Appendix II includes postulates for equality and the definitions of subtraction and division. In Chapter 3 we discussed some properties of order and defined positive and negative. The properties of order may be considered as our postulates for inequality. Appendix III includes several theorems which extend the discussion in Chapter 3 concerning the order properties of real numbers.

The addition property of order as given in the text can be extended as in Theorem III-1.

THEOREM III-1. For all x, y, a, b, if $x > y$ and $a > b$ then $x + a > y + b$.

Proof:

$x > y$	hypothesis
$x + a > y + a$	additive property of order
$a > b$	hypothesis
$a + y > b + y$	additive property of order
$y + a > y + b$	commutative property of addition
$x + a > y + b$	trans. property of order

The following theorem shows that if we accept the first part of the multiplication property of order as given in the text we can prove the second part.

THEOREM III-2. For all x, y, k, if $x > y$ and $k < 0$ then $kx < ky$.

Proof:

$0 > k$	hypothesis
$0 + (-k) > k + (-k)$	additive property of order
$0 + (-k) > 0$	additive inverse
$-k > 0$	additive identity
$x > y$	hypothesis
$(-k)x > (-k)y$	mult. property of order, Part 1
$-kx > -ky$	Theorem II-3
$-kx + (kx + ky) > -ky + (kx + ky)$	additive property for order
$-kx + (ky + kx) > -ky + (kx + ky)$	commutative prop. for add.
$ky > kx$	Theorem II-8

Theorems III-3 and III-4 are precise statements concerning the comparison of two real numbers.

THEOREM III-3. For all x and y, $x > y$ if and only if $x - y$ is positive.

(There are two parts to prove.)

Part 1. If $x - y$ is positive then $x > y$.

Proof:

$x - y > 0$	definition of positive
$(x - y) + y > 0 + y$	additive property of order
$y + (x - y) > y + 0$	commutative law for addition
$x > y + 0$	Theorem II-8
$x > y$	additive identity

Part 2. If $x > y$ then $x - y$ is positive.

Proof:

$x > y$	hypothesis
$x + (-y) > y + (-y)$	additive property of order
$x + (-y) > 0$	additive inverse
$x - y > 0$	definition of subtraction
$x - y$ is positive	definition of positive

THEOREM III-4. For all x and y, $x > y$ if and only if there is a number z such that $z > 0$ and $x = y + z$.

(There are two parts to prove.)

Part 1. If $x = y + z$ and $z > 0$ then $x > y$.

Proof:

$x = y + z$	hypothesis
$z = x - y$	Theorem II-5
$x - y$ is positive	hypothesis
$x > y$	Theorem III-3

Part 2. If $x > y$ then there is a positive number z such that $y + z = x$.

Proof:

$x > y$	hypothesis
$x - y$ is positive	Theorem III-3
$y + (x - y) = x$	Theorem II-8

Observe that $x - y$ is a positive number with the required property.

Appendix IV

RATIONAL AND IRRATIONAL NUMBERS

How to Show That a Number is Rational.

By definition a number is rational if it is the quotient of two integers. Therefore, if we want to prove that a number x is rational, we have to show that there are two integers p and q, such that $\frac{p}{q} = x$. Here are some examples:

(1) The number $x = \frac{1}{2} + \frac{3}{7}$ is rational, because

$$\frac{1}{2} + \frac{3}{7} = \frac{7 + 6}{14} = \frac{13}{14} .$$

Therefore $x = \frac{p}{q}$, where $p = 13$ and $q = 14$.

(2) The number $x = 1.23$ is rational, because

$$1.23 = \frac{123}{100} ,$$

which is the quotient of the two integers 123 and 100.

(3) If the number x is rational, then so is the number $2x$. (That is, twice a rational number is always rational.) For if

$$x = \frac{p}{q} ,$$

where p and q are integers, then

$$2x = \frac{2p}{q} ,$$

where the numerator $2p$ and the denominator q are both integers.

(4) If the number x is rational, then so is the number $x + \frac{2}{3}$. For if

$$x = \frac{p}{q} ,$$

then

$$x + \frac{2}{3} = \frac{p}{q} + \frac{2}{3} = \frac{3p + 2q}{3q} .$$

where the numerator and denominator are both integers.

(5) If x is a rational number, then so is $x^2 + x$. For if

$$x = \frac{p}{q},$$

then

$$x^2 + x = \frac{p^2}{q^2} + \frac{p}{q} = \frac{p^2 + pq}{q^2}$$

where the numerator and denominator are integers.

Problem Set IV-1

1. Show that .2351 is a rational number.
2. Show that $\frac{2}{3} + \frac{5}{7}$ is rational.
3. Show that if x is a rational number, then so is $x - 5$.
4. Show that if x is rational, then so is $2x - 7$.
5. Show that $\frac{1}{3} + \frac{1}{17}$ is rational.
6. Show that the sum of _any_ two rational numbers is a rational number.
7. Show that $(\frac{17}{19})(\frac{23}{47})$ is rational.
8. Show that the product of _any_ two rational numbers is a rational number.
9. Show that $\frac{3}{17} + \frac{23}{7}$ is rational.
10. Show that the quotient of _any_ two rational numbers is a rational number, as long as the divisor is not zero.
11. Given that $\sqrt{2}$ is irrational, show that $\frac{\sqrt{2}}{2}$ is also irrational. (Hint: This problem is a lot easier, now that you understand about indirect proofs.)
12. Given that π is irrational, show that $\frac{\pi}{5}$ is also irrational.
13. Show that the reciprocal of every rational number different from zero is rational.
14. Show that the reciprocal of every irrational number different from zero is irrational.

15. Is it true that the sum of a rational number and an irrational number is always irrational? Why or why not?

16. Is it true that the sum of two irrational numbers is always irrational? Why or why not?

17. How about the product of a rational number and an irrational number?

Some Examples of Irrational Numbers.

In the previous section, we proved that under certain conditions a number must be rational. In some of the problems, you showed that starting with an irrational number we could get more irrational numbers in various ways. In all this we left one very important question unsettled: are there any irrational numbers? We shall settle this question by showing that a particular number, namely $\sqrt{2}$ cannot be expressed as the ratio of any two integers.

To prove this, we first need to establish some of the facts about squares of odd and even integers. Every integer is either even or odd. If n is even, then n is twice some integer k, and we can write

$$n = 2k \ .$$

If n is odd, then when we divide by 2 we get a quotient k and a remainder 1, so that

$$\frac{n}{2} = k + \frac{1}{2} \ .$$

Therefore, we can write

$$n = 2k + 1 \ .$$

These are the typical formulas for even numbers and odd numbers, respectively. For example,

$$\begin{array}{ll} 6 = 2 \cdot 3 & n = 6 \ , \ k = 3 \\ 7 = 2 \cdot 3 + 1 & n = 7 \ , \ k = 3 \\ 8 = 2 \cdot 4 & n = 8 \ , \ k = 4 \\ 9 = 2 \cdot 4 + 1 & n = 9 \ , \ k = 4 \end{array}$$

and so on. The following theorem is easy to prove:

THEOREM IV-1. The square of every odd number is odd.

Proof: If n is odd, then we can write

$$n = 2k + 1 ,$$

where k is an integer. Squaring both sides, we get

$$\begin{aligned} n^2 &= (2k)^2 + 2 \cdot 2k + 1 \\ &= 4k^2 + 4k + 1 . \end{aligned}$$

The right-hand side must be odd, because it is written in the form

$$2 \cdot [2k^2 + 2k] + 1 ;$$

that is, it is twice an integer, plus 1. Therefore, n^2 is odd, which was to be proved.

From Theorem IV-1 we can quickly get another theorem:

THEOREM IV-2. If n^2 is even, then n is even.

Proof: If n were odd, then n^2 would be odd, which is false. Therefore n is even.

Notice that is is an indirect proof.

We are now ready to begin the proof of:

THEOREM IV-3. $\sqrt{2}$ is irrational.

Proof: The proof will be indirect. We begin by making the assumption that $\sqrt{2}$ is rational. We will show that this leads to a contradiction.

Step 1. Supposing that $\sqrt{2}$ is rational, it follows that $\sqrt{2}$ can be expressed as

$$\sqrt{2} = \frac{p}{q}$$

where the fraction $\frac{p}{q}$ is in lowest terms.

The reason is that if $\sqrt{2}$ can be expressed as a fraction at all, then we can reduce the fraction to lowest terms by dividing out any common factors of the numerator and denominator.

We therefore have

$$\sqrt{2} = \frac{p}{q} ,$$

in lowest terms. This gives

$$2 = \frac{p^2}{q^2} ,$$

which in turn gives

$$p^2 = 2q^2 .$$

Step 2. p^2 is even,
because p^2 is twice an integer.

Step 3. p is even,
by Theorem IV-2.

We therefore set $p = 2k$. Substituting in the formula at the end of Step 1, we get

$$(2k)^2 = 2q^2 ,$$

which means that

$$4k^2 = 2q^2 .$$

Therefore

$$q^2 = 2k^2 .$$

Step 4. q^2 is even,
because q^2 is twice an integer.

Step 5. q is even,
by Theorem IV-2.

We started by assuming that $\sqrt{2}$ was rational. From this we got $\sqrt{2} = \frac{p}{q}$, in lowest terms. From this we have proved that p and q were both even. Therefore $\frac{p}{q}$ was not in lowest terms, after all. This contradiction shows that our initial assumption must have been wrong, that is, $\sqrt{2}$ must not be rational.

Problem Set IV-2

These problems are harder than most of the problems in the text.

1. Adapt the proof that $\sqrt{2}$ is irrational, so as to get a proof that $\sqrt{3}$ is irrational. (Hint: Start with the fact that every integer has one of the forms

$$n = 3k$$
$$n = 3k + 1$$
$$n = 3k + 2 ,$$

and then prove a theorem corresponding to Theorem IV-2.)

2. Obviously nobody can prove that $\sqrt{4}$ is irrational, because $\sqrt{4} = 2$. If you try to "prove" by adapting the proof for $\sqrt{2}$, at what point does the "proof" break down?

3. Show that $\sqrt[3]{2}$ is irrational.

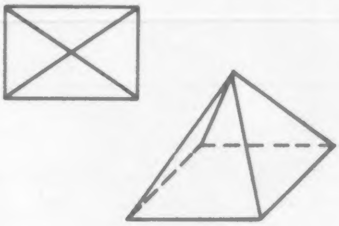
Actually, the square root of an integer is either another integer or an irrational number; that is, $\sqrt{n}$ either "comes out very even" or "comes out very uneven." The proof of this fact, however, requires more mathematical technique than we now have at our disposal. Problems like this are solved in a branch of mathematics called the Theory of Numbers.

Appendix V

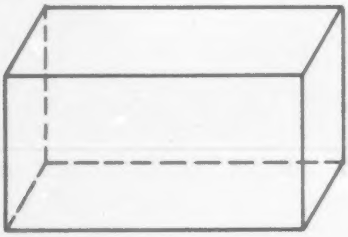
HOW TO DRAW PICTURES OF SPACE FIGURES

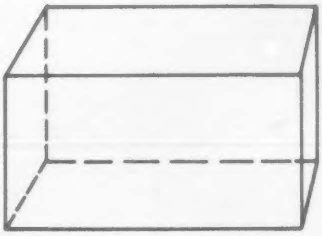
Simple Drawing.

A course in mechanical drawing is concerned with precise representation of physical objects seen from different positions in space. In geometry we are concerned with drawing only to the extent that we use sketches to help us do mathematical thinking. There is no one correct way to draw pictures in geometry, but there are some techniques helpful enough to be in rather general use. Here, for example, is a technically correct drawing of an ordinary pyramid, for a person can argue that he is looking at the pyramid from directly above. But careful ruler drawing is not as helpful as this very crude free-hand sketch. The first drawing does not suggest 3-space; the second one does.



The first part of this discussion offers suggestions for simple ways to draw 3-space figures. The second part introduces the more elaborate technique of drawing from perspective. The difference between the two approaches is suggested by these two drawings of a rectangular box.

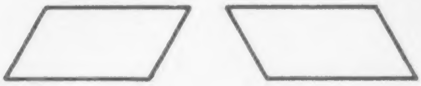




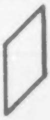
In the first drawing the base is shown by an easy-to-draw parallelogram. In the second drawing, the front base edge and the back base edge are parallel, but the back base edge is drawn shorter under the belief that the shorter length will suggest "more remote."

No matter how a rectangular box is drawn, some sacrifices must be made. All angles of a rectangular solid are right angles, but in each of the drawings shown on the previous page two-thirds of the angles do not come close to indicating ninety degrees when measured with a protractor. We are willing to give up the drawing of right angles that look like right angles in order that we make the figure as a whole more suggestive.

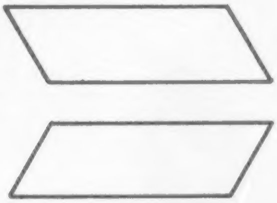
You already know that a plane is generally pictured by a parallelogram.

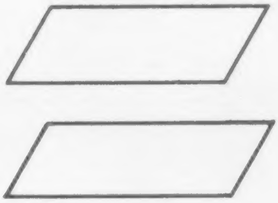


It seems reasonable to draw a horizontal plane in either of the ways shown, and to draw a vertical plane like this.

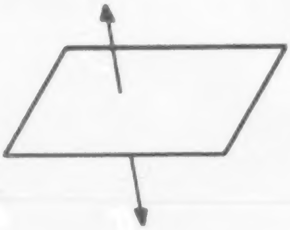


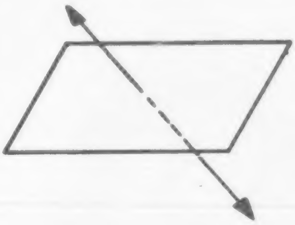
If we want to indicate two parallel planes, however, we can not be effective if we just draw any two "horizontal" planes. Notice how the drawing the right below improves upon the one to the left. Perhaps you prefer still another kind of drawing.



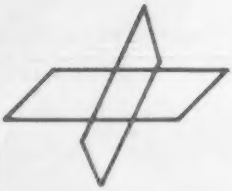


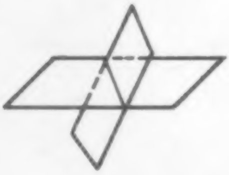
Various devices are used to indicate that one part of a figure passes behind another part. Sometimes a hidden part is simply omitted, sometimes it is indicated by dotted lines. Thus, a line piercing a plane may be drawn in either of the two ways:

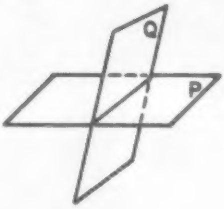


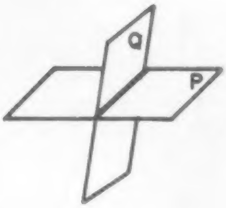


Two intersecting planes are illustrated by each of these drawings.

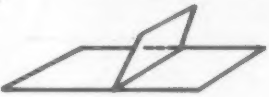






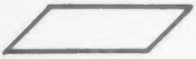


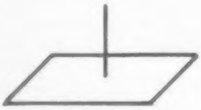
The second is better than the first because the line of intersection is shown and parts concealed from view are dotted. The third and fourth drawings are better yet because the line of intersection is visually tied in with plane P as well as plane Q by the use of parallel lines in the drawing. Here is a drawing which has the advantage of simplicity and the disadvantage of suggesting one plane and one halfplane.

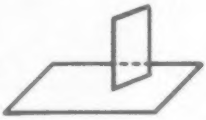


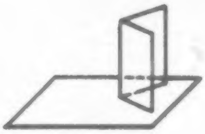
In any case a line of intersection is a particularly important part of a figure.

Suppose that we wish to draw two intersecting planes each perpendicular to a third plane. An effective procedure is shown by this step-by-step development.

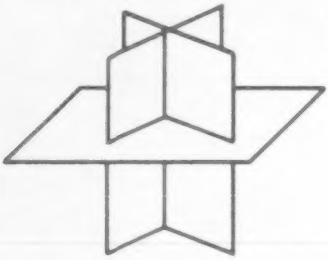


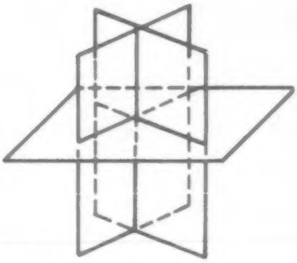




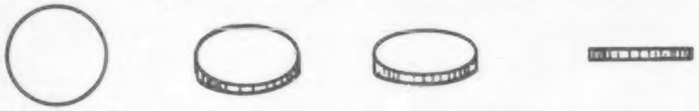


Notice how the last two planes drawn are built on the line of intersection. A complete drawing showing all the hidden lines is just too involved to handle pleasantly. The picture below is much more suggestive.

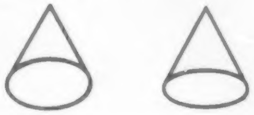




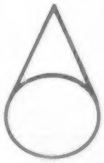
A dime, from different angles, looks like this:



Neither the first nor the last is a good picture of a circle in 3-space. Either of the others is satisfactory. The thinner oval is perhaps better to use to represent the base of a cone.

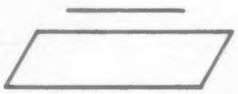


Certainly nobody should expect us to interpret the figure shown below as a cone.

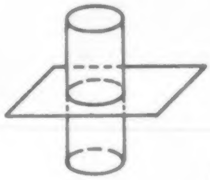


A few additional drawings, with verbal descriptions, are shown.

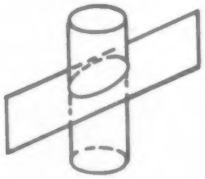
A line parallel to a plane.



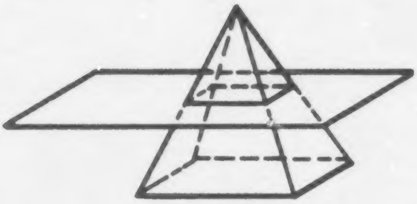
A cylinder cut by a plane parallel to the base.



A cylinder cut by a plane not parallel to the base.



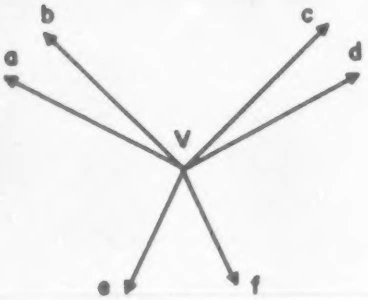
A pyramid cut by a plane
parallel to the base.

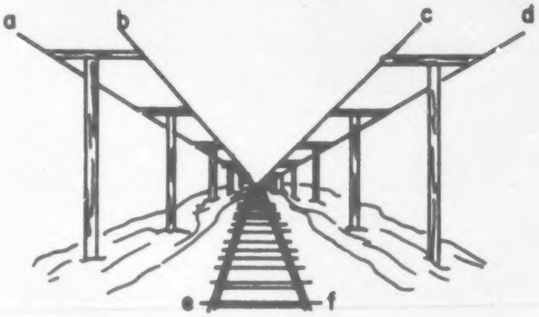


It is important to remember that a drawing is not an end in itself but simply an aid to our understanding of the geometrical situation. We should choose the kind of picture that will serve us best for this purpose, and one person's choice may be different from another.

Perspective.

The rays a , b , c , d , e , f in the left-hand figure below suggest coplanar lines intersecting at V ; the corresponding rays in the right-hand figure suggest parallel lines in a three-dimensional drawing. Think of a railroad track and telephone poles as you look at the right-hand figure.





The right-hand figure suggests certain principles which are useful in making perspective drawings.

(1) A set of parallel lines which recede from the viewer are drawn as concurrent rays; for example, rays a , b , c , d , e , f . The point on the drawing where the rays meet is known as the "vanishing point."

(2) Congruent segments are drawn smaller when they are farther from the viewer. (Find examples in the drawing.)

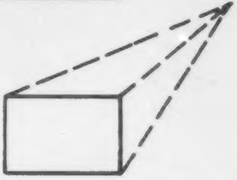
(3) Parallel lines which are perpendicular to the line of sight of the viewer are shown as parallel lines in the drawing. (Find examples in the drawing.)

A person does not need much artistic ability to make use of these three principles.

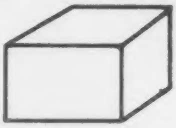
The steps to follow in sketching a rectangular solid are shown below.

Draw the front face as a rectangle.

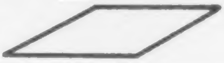
Select a vanishing point and draw segments from it to the vertices. Omit segments that cannot be seen.



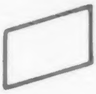
Draw edges parallel to those of the front face. Finally erase lines of perspective.



Under this technique a single horizontal plane can be drawn as the top face of the solid shown above.



A single vertical plane can be represented by the front face or the right-hand face of the solid.



After this brief account of two approaches to the drawing of figures in 3-space we should once again recognize the fact that there is no one correct way to picture geometric ideas. However, the more "real" we want our picture to appear, the more attention we should pay to perspective. Such an artist as Leonardo da Vinci paid great attention to perspective. Most of us find this done for us when we use ordinary cameras.

See some books on drawing or look up "perspective" in an encyclopedia if you are interested in a detailed treatment.

Appendix VI

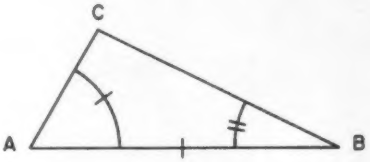
THE A.S.A. AND S.S.S. CONGRUENCE POSTULATES AS THEOREMS

In Chapter 5 we assume three postulates about congruence of triangles - S.A.S.; A.S.A., and S.S.S. We stated that two of these are "redundant"; i.e., it is unnecessary to assume them, for they can be proved. (They were assumed in Chapter 5 to expedite and simplify our development of congruence.)

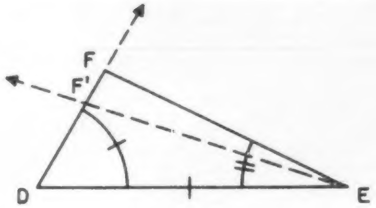
We now proceed to prove the A.S.A. and S.S.S. statements, to show that proofs really can be given. We still assume the S.A.S. Postulate.

THEOREM VI-1. (A.S.A.) Given a correspondence between two triangles (or between a triangle and itself). If two angles and the included side of one triangle are congruent to the corresponding parts of the other triangle, then the correspondence is a congruence.

Proof: Let $ABC \longleftrightarrow DEF$ be the correspondence between the triangles. We have as hypothesis: $\angle A \cong \angle D$, $AB = DE$, $\angle B \cong \angle E$; and we are required to prove that $\triangle ABC \cong \triangle DEF$.



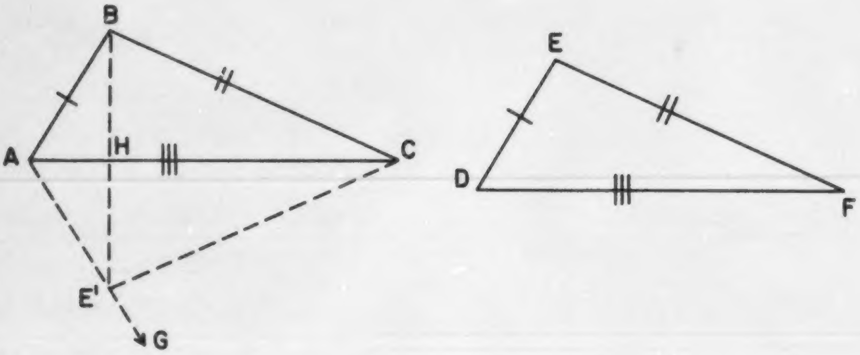
We proceed as follows:



Statements	Reasons
1. On the ray $\overrightarrow{DF}$ there is a point F', such that $DF' = AC$.	1. Point Plotting Theorem.
2. $AB = DE$ and $m\angle A = m\angle D$.	2. Hypothesis.
3. $\triangle ABC \cong \triangle DEF'$.	3. S.A.S. Postulate.
4. $\angle ABC \cong \angle DEF'$.	4. Definition of a congruence between triangles.
5. $\angle ABC \cong \angle DEF$.	5. Hypothesis.
6. $\angle DEF' \cong \angle DEF$.	6. Transitive Property of Congruence for Angles.
7. $\overrightarrow{EF'}$ and $\overrightarrow{EF}$ are the same ray.	7. Protractor Postulate.
8. $F' = F$.	8. Two lines intersect in at most one point.
9. $\triangle ABC \cong \triangle DEF$.	9. Substitution Property of Equality.

THEOREM VI-2. (S.S.S.) Given a correspondence between two triangles (or between a triangle and itself). If all three pairs of corresponding sides are congruent, then the correspondence is a congruence.

Proof: Let $ABC \longleftrightarrow DEF$ be the correspondence between the triangles. We have as hypothesis: $AB = DE$; $BC = EF$; $CA = FD$; and we are to prove that $\triangle ABC \cong \triangle DEF$.



We proceed as follows:

Statements	Reasons
1. There is a ray $\overrightarrow{AG}$, such that $\angle CAG \cong \angle FDE$, and such that B and G are on opposite sides of $\overleftrightarrow{AC}$.	1. Protractor Postulate.
2. There is a point E' on $\overrightarrow{AG}$, such that $AE' = DE$.	2. Point Plotting Theorem.
3. $\Delta AE'C \cong \Delta DEF$.	3. S.A.S. Postulate.

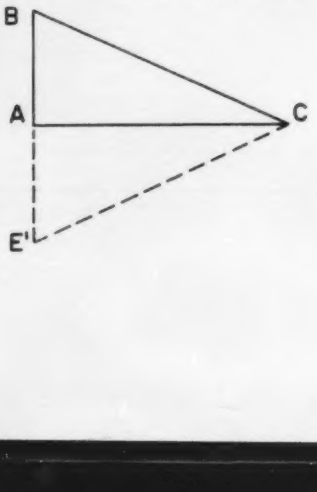
What we have done, so far, is to duplicate ΔDEF on the under side of ΔABC, using the S.A.S. Postulate.

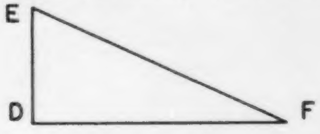
Statements	Reasons
4. $AB = DE$.	4. Hypothesis.
5. $DE = AE'$.	5. Step 2.
6. $AB = AE'$.	6. Transitive Property of Equality.
7. $BC = EF$.	7. Hypothesis.
8. $EF = E'C$.	8. Corresponding parts of congruent triangles are congruent.
9. $BC = E'C$.	9. Transitive Property of Equality.
10. $\overline{BE'}$ intersects $\overline{AC}$ in a point H.	10. Plane Separation Postulate.

We shall now complete the proof for the case in which H is between A and C, as in the figure. The other possible cases will be discussed later.

11. $\overrightarrow{BH}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BC}$.	11. Interior of an Angle Postulate.
12. $\overline{AB} = \overline{AE'}$.	12. Segments of equal length are congruent.
13. $\triangle ABE'$ is isosceles.	13. A triangle with two sides congruent is isosceles.
14. $\angle ABH \cong \angle AE'H$.	14. Isosceles Triangle Theorem.
15. $\angle CBH \cong \angle CE'H$.	15. Isosceles Triangle Theorem.
16. $m\angle ABH + m\angle CBH = m\angle ABC$.	16. The Betweenness-Angles Theorem.
17. $m\angle AE'H + m\angle CE'H = m\angle AE'C$.	17. The Betweenness-Angles Theorem.
18. $m\angle ABC = m\angle AE'C$.	18. The Substitution Property for Equality.
19. $\angle ABC \cong \angle AE'C$.	19. Definition of Congruent Angles.
20. $\angle ABC \cong \angle DEF$.	20. Transitive Property of Congruence for Angles.
21. $\triangle ABC \cong \triangle DEF$.	21. S.A.S. Postulate.

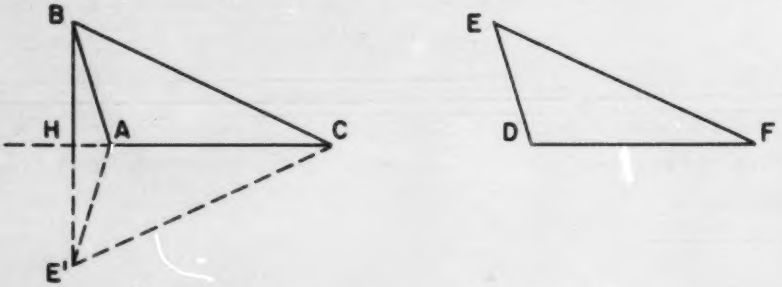
This completes the proof for the case in which H is between A and C. We recall that H is the point in which the line $\overleftrightarrow{BE'}$ intersects the line $\overleftrightarrow{AC}$. If H = A, then B, A and E' are collinear, and the figure looks like this:





In this case $\angle B \cong \angle E'$, because the base angles of an isosceles triangle are congruent. Therefore $\angle B \cong \angle E$, because $\angle E \cong \angle E'$. The S.A.S. Postulate applies, as before, to show that $\Delta ABC \cong \Delta DEF$.

If A is between H and C , then the figure looks like this:



and we show that $\angle ABC \cong \angle E$ by subtracting the measures of angles, instead of by adding them. That is,

$$m \angle ABC = m \angle HBC - m \angle HBA$$

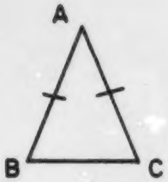
and $$m \angle AE'C = m \angle HE'C - m \angle HE'A ,$$

so that $$\angle ABC \cong \angle AE'C \cong \angle DEF ,$$

as before. The rest of the proof is the same as in the first case.

The two remaining cases, H = C and C between A and H , are similar to the two above.

In the proof of this theorem we used the Isosceles Triangle Theorem. The proof of this theorem, as given in Chapter 5, depended on the S.S.S. Postulate. Therefore, as matters stand, we could be convicted of circular reasoning. This circle of reasoning can be broken easily by observing that a proof of the Isosceles Triangle Theorem can be based on the S.A.S. Postulate rather than the S.S.S. Postulate. Such an alternate proof is the following one.



Suppose given triangle $\triangle ABC$ with $\overline{AB} \cong \overline{AC}$. Then $\overline{AB} \cong \overline{AC}$, $\overline{AC} \cong \overline{AB}$, $\angle A \cong \angle A$, and by the S.A.S. Postulate, the correspondence $ABC \longleftrightarrow ACB$ is a congruence. Consequently $\angle B \cong \angle C$, and the Isosceles Triangle Theorem is proved.

With this remark about the proof of the Isosceles Triangle Theorem we have completed our task of showing that the A.S.A. and S.S.S. Postulates can be deduced as theorems from the S.A.S. Postulate.

The Meaning and Use of Symbols

General.

$=$. $A = B$ can be read as "A equals B", "A is equal to B", "A equal B" (as in "Let $A = B$"), and possibly other ways to fit the structure of the sentence in which the symbol appears. However, we should not use the symbol, $=$, in such forms as "A and B are $=$"; its proper use is *between two expressions*. If two expressions are connected by "$=$" it is to be understood that these two expressions stand for the same mathematical entity, in our case either a real number of a point set.

$\neq$. "Not equal to". $A \neq B$ means that A and B do *not* represent the same entity. The same variations and cautions apply to the use of $\neq$ as to the use of $=$.

Algebraic.

$+$, $\cdot$, $-$, $\div$. These familiar algebraic symbols for operating with real numbers need no comment. The basic postulates about them are presented in Appendix II.

$<$, $>$, $\leq$, $\geq$. Like $=$, these can be read in various ways in sentences, and $A < B$ may stand for the underlined part of "If *A is less than B*" , "Let *A be less than B*" , "*A less than B* implies", etc. Similarly for the other three symbols, read "greater than", "less than or equal to", "greater than or equal to". These inequalities apply only to real numbers. Their properties are discussed in Chapter 3 and Appendix III.

Geometric.

Point Sets. A single letter may stand for any point set. Thus we may speak of a point P , a line m , a halfplane $\mathcal{H}$, a circle C , an angle x , a segment b , etc.

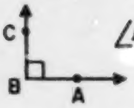
$\overleftrightarrow{AB}$. The line containing the two points A and B .

$\overline{AB}$. The segment having A and B as endpoints.

$\overrightarrow{AB}$. The ray with A as its endpoint and containing point B .

$\angle ABC$. The angle having B as vertex and $\overrightarrow{BA}$ and $\overrightarrow{BC}$ as sides.

$\triangle ABC$. The triangle having A, B, C as vertices.



$\angle ABC$ is a right angle.

$\angle A\text{-}BC\text{-}D$. The dihedral angle having line $\overleftrightarrow{BC}$ as edge and with sides containing A and D .

Real Numbers.

AB. The positive number which is the distance between the two points A and B , and also the length of the segment $\overline{AB}$.

m $\angle ABC$. The real number between 0 and 180 which is the degree measure of $\angle ABC$.

PQ (relative to $[A,A']$). The measure of the segment $\overline{PQ}$, with respect to the unit-pair, $[A,A']$.

Relations.

$a \longleftrightarrow a'$. a is matched with a' .

$\cong$. Congruence. $A \cong B$ is read "A is congruent to B", but with the same possible variations and restrictions as $A = B$. In the text A and B may be segments, angles, or triangles.

$\perp$. Perpendicular. $A \perp B$ is read "A is perpendicular to B", with the same comment as for $\cong$. A and B may be either two lines, or subsets of lines (rays or segments).

The Greek Alphabet

A	α	alpha	a	(a)	N	ν	nu	n	(n)
B	β	beta	b	(b)	Ξ	ξ	xi	x	(ks)
Γ	γ	gamma	g	(g)	O	ο	omicron	o	(o)
Δ	δ	delta	d	(d)	π	π	pi	p	(p)
E	ϵ	epsilon	e	(e)	P	ρ	rho	r, rh	(r)
Z	ζ	zeta	z	(z)	Σ	σ	sigma	s	(s)
H	η	eta	e	(a)	T	τ	tau	t	(t)
θ	θ	theta	th	(th)	ϒ	υ	upsilon	y, u	(u, oo)
I	ι	iota	i	(e)	Φ	ϕ	phi	ph	(f)
K	κ	kappa	k	(k)	X	χ	chi	ch	(k, K)
Λ	λ	lambda	l	(l)	Ψ	ψ	psi	ps	(ps)
M	μ	mu	m	(m)	Ω	ω	omega	o	(o)

List of Chapters and Postulates

Chapter 1. Introduction to Formal Geometry

Chapter 2. Sets; Points, Lines and Planes

Postulate 1. Space contains at least two distinct points.

Postulate 2. Every line is a set of points and contains at least two points.

Postulate 3. If P and Q are two distinct points, there is one and only one line that contains them.

Postulate 4. No line contains all points of space.

Postulate 5. Every plane is a set of points and contains at least three noncollinear points.

Postulate 6. If P, Q, R are three distinct noncollinear points, then there is one and only one plane which contains them.

Postulate 7. No plane contains all points of space.

Postulate 8. If two distinct points of a line belong to a plane, then every point of the line belongs to that plane.

Postulate 9. If two planes intersect, then their intersection is a line.

Chapter 3. Distance and Coordinate Systems

Postulate 10. If A and A' are distinct points, there exists a correspondence which associates with each pair of distinct points in space a unique positive number such that the number assigned to the given pair of points $\{A, A'\}$ is one.

Postulate 11. If $\{A, A'\}$ is any unit-pair and if B and B' are two points such that

$$BB' \text{ (relative to } \{A, A'\}) = 1, \text{ then}$$

for any pair of points, the distance between them relative to the unit-pair $\{B, B'\}$ is the same as the distance between them relative to $\{A, A'\}$.

Postulate 12. (The Ruler Postulate) If $\{A,A'\}$ is any unit-pair, if ℓ is any line, and if P and Q are any two distinct points on ℓ, then there is a unique coordinate system on ℓ relative to $\{A,A'\}$ such that the origin of the coordinate system is P and the coordinate of Q is positive.

Postulate 13. Let A and A' be any two distinct points and let B and B' be any two distinct points. Then, for every pair of distinct points P and Q in space, $\dfrac{PQ \text{ (relative to } \{A,A'\})}{PQ \text{ (relative to } \{B,B'\})}$ is a constant.

Chapter 4. Angles

Postulate 14. (The Plane Separation Postulate) For any plane and any line contained in the plane, the points of the plane which do not lie on the line form two sets such that

(1) each of the two sets is convex, and

(2) every segment which joins a point of one of the sets and a point of the other intersects the given line.

Postulate 15. For any plane, the points of space which do not lie on the plane form two sets such that

(1) each of the two sets is convex, and

(2) every segment which joins a point of one of the sets and a point of the other intersects the given plane.

Postulate 16. There exists a correspondence which associates with each angle in space a unique number between 0 and 180.

Postulate 17. (The Protractor Postulate) If $\mathcal{F}$ is any plane and if $\overrightarrow{VA}$ and $\overrightarrow{VB}$ are noncollinear rays in $\mathcal{F}$, then there is a unique ray-coordinate system in $\mathcal{F}$ relative to V such that $\overrightarrow{VA}$ corresponds to 0 and such that every ray $\overrightarrow{VX}$ with X and B on the same side of $\overleftrightarrow{VA}$ corresponds to a number less than 180.

Postulate 18. (The Interior of an Angle Postulate)
If $\angle AVB$ is any angle,

(1) Let $\mathcal{R}$ be the set of all interior points of rays between $\overrightarrow{VA}$ and $\overrightarrow{VB}$,

(2) Let $\mathcal{U}$ be the set of all points which belong both to the halfplane with edge $\overleftrightarrow{VA}$ and containing B and to the halfplane with edge $\overleftrightarrow{VB}$ and containing A, and

(3) Let $\mathcal{S}$ be the set of all interior points of segments joining an interior point of $\overrightarrow{VA}$ and an interior point of $\overrightarrow{VB}$.

Then $\mathcal{R}$ and $\mathcal{U}$ are the same set, and this set contains $\mathcal{S}$.

Chapter 5. Congruence

Postulate 19. (The S.A.S. Postulate) Given a one-to-one correspondence between the vertices of two triangles (not necessarily distinct). If two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

Postulate 20. (The A.S.A. Postulate) Given a one-to-one correspondence between the vertices of two triangles (not necessarily distinct). If two angles and the included side of one triangle are congruent to the corresponding parts of the other triangle, then the correspondence is a congruence.

Postulate 21. (The S.S.S. Postulate) Given a one-to-one correspondence between the vertices of two triangles (not necessarily distinct). If three sides of one triangle are congruent to the corresponding sides of the other triangle, then the correspondence is a congruence.

CHAPTER 6. Parallelism

Postulate 22. (The Parallel Postulate) There is at most one line parallel to a given line and containing a given point not on the given line.

CHAPTER 7. Similarity

Postulate 23. (The Proportional Segments Postulate) If a line is parallel to one side of a triangle and intersects the other two sides in interior points, then the measures of one of those side and the two segments into which it is cut are proportional to the measures of the three corresponding segments in the other side.

LIST OF THEOREMS AND COROLLARIES

THEOREM 2-1. Space contains at least one line.

THEOREM 2-2. Space contains at least three distinct points not in one line.

THEOREM 2-3. Space contains at least two lines.

THEOREM 2-4. If two distinct lines intersect, they intersect in exactly one point.

THEOREM 2-5. If P is a point, there is a plane that contains it.

THEOREM 2-6. Space contains at least two planes.

THEOREM 2-7. Space contains at least four noncoplanar points.

THEOREM 2-8. If a line intersects a plane not containing it, the intersection is a single point.

THEOREM 2-9. A line and a point not on that line are contained in exactly one plane.

THEOREM 2-10. If two distinct lines have a point in common, there is exactly one plane which contains them.

THEOREM 3-1. (The Origin and Unit-Point Theorem) If P and Q are any two distinct points, then there is a coordinate system on the line $\overleftrightarrow{PQ}$ relative to the unit-pair $\{P,Q\}$ such that P is the origin and Q is the unit-point of the coordinate system.

THEOREM 3-2. Every point on a given line is the endpoint of two rays on the line, and the intersection of these two rays is the point itself.

THEOREM 3-3. Every segment has a unique midpoint.

THEOREM 3-4. Let $\{A,A'\}$ and $\{B,B'\}$ be any unit-pairs, let M and N be any two distinct points, and let E and F be any two distinct points. Then

$$\frac{MN(\text{relative to } \{A,A'\})}{EF(\text{relative to } \{A,A'\})} = \frac{MN(\text{relative to } \{B,B'\})}{EF(\text{relative to } \{B,B'\})}$$

THEOREM 3-5. (The Two Coordinate System Theorem) Let a line ℓ and two coordinate systems, C and C', on ℓ be given. There exist two numbers a, b, with $a \neq 0$, such that for any point on ℓ, its coordinate x in C is related to its coordinate x' in C' by the equation $x' = ax + b$.

THEOREM 3-6. (The Two-Point Theorem) In any coordinate system on a line ℓ, let x_1 and x_2 be the respective coordinates of distinct points X_1 and X_2 on ℓ. Then the formula

$$x = x_1 + k(x_2 - x_1)$$

expresses the coordinate x of any point on ℓ in terms of the coordinate k of the same point relative to the coordinate system with origin X_1 and unit-point X_2.

THEOREM 3-7. (The Betweenness-Coordinates Theorem) Let C, D, F be three points on a line ℓ and let any coordinate system on ℓ be given. The point F is between the points C and D if and only if the coordinate of F is between the coordinates of C and of D.

THEOREM 3-8. (The Point Plotting Theorem) Let $\{A,A'\}$ be any unit-pair, let Q be any point, and let p be any positive number. On any ray with endpoint Q there is a unique point R such that the distance QR is p.

THEOREM 3-9. (The Betweenness-Distance Theorem) Let B, C, D be points such that C is between B and D. If $\{A,A'\}$ is any unit-pair, then the distances relative to $\{A,A'\}$ satisfy the condition that $BC + CD = BD$ (or, that $BC = BD - CD$.)

THEOREM 4-1. The intersection of any two convex sets of points is a convex set.

THEOREM 4-2. If the intersection of a line and a ray is the endpoint of the ray, then the interior of the ray is contained in one of the halfplanes whose edge is the given line.

THEOREM 4-3. (Angle Construction Theorem) If $\mathcal{H}$ is a halfplane whose edge contains the ray $\overrightarrow{VA}$ and if r is any number between 0 and 180, then there is a unique ray $\overrightarrow{VR}$ such that R is in $\mathcal{H}$ and $m\angle AVR = r$.

THEOREM 4-4. (The Betweenness-Angles Theorem) Let $\overrightarrow{VE}$, $\overrightarrow{VF}$, $\overrightarrow{VG}$ be rays such that $\overrightarrow{VF}$ is between $\overrightarrow{VE}$ and $\overrightarrow{VG}$. Then $m\angle EVF + m\angle FVG = m\angle EVG$ (or, $m\angle EFV = m\angle EVG - m\angle FVG$.)

THEOREM 4-5. Every angle has a unique midray.

THEOREM 4-6. The interior of any angle is a convex set.

THEOREM 4-7. Let A, B, C, E, F, G be coplanar points such that A, B, C are not collinear, E is between B and C, the rays $\overrightarrow{EF}$ and $\overrightarrow{EA}$ are opposite, and the rays $\overrightarrow{CG}$ and $\overrightarrow{CA}$ are opposite. Then $\overrightarrow{CF}$ is between $\overrightarrow{CB}$ and $\overrightarrow{CG}$.

THEOREM 4-8. The sum of the measures of the two angles in any linear pair is 180.

THEOREM 4-9. Let A, B, O, X, Y be distinct coplanar points such that O is between X and Y, such that A and B are on the same side of $\overleftrightarrow{XY}$, and such that $\overrightarrow{OA}$ is between $\overrightarrow{OX}$ and $\overrightarrow{OB}$. Then $m\angle XOA + m\angle AOB + m\angle BOY = 180$.

THEOREM 4-10. Two adjacent angles, such that the sum of their measures is 180, are a linear pair of angles.

THEOREM 4-11. If the two angles of a linear pair have the same measure, then each of them is a right angle.

THEOREM 4-12. Any two right angles are congruent to each other.

THEOREM 4-13. (The Supplement Theorem) The two angles of any linear pair are supplementary to each other.

THEOREM 4-14. If two angles are both congruent and supplementary, then each of them is a right angle.

THEOREM 4-15. If two angles are complementary, then each of them is acute.

THEOREM 4-16. Supplements of congruent angles are congruent to each other.

THEOREM 4-17. Complements of congruent angles are congruent to each other.

THEOREM 4-18. Let each of two sets be a line or a ray or a segment. If the two lines which are determined, respectively, by the given sets intersect in a single point V, then the given sets determine two pairs of vertical angles, all with vertex V.

THEOREM 4-19. Any two vertical angles are congruent to each other.

THEOREM 4-20. If two intersecting lines form one right angle, then they form four right angles.

THEOREM 4-21. For each point on a line in a plane, there is one and only one line which lies in the given plane, contains the given point, and is perpendicular to the given line.

THEOREM 4-22. The interior of a triangle is a convex set.

THEOREM 4-23. The interior of any convex polygon is a convex set.

THEOREM 5-1. Congruence for segments has the following properties:

Reflexive: $\overline{AB} \cong \overline{AB}$.

Symmetric: If $\overline{AB} \cong \overline{CD}$ then $\overline{CD} \cong \overline{AB}$

Transitive: If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$.

THEOREM 5-2. Congruence for angles has the following properties:

Reflexive: $\angle A \cong \angle A$.

Symmetric: If $\angle A \cong \angle B$ then $\angle B \cong \angle A$.

Transitive: If $\angle A \cong \angle B$ and $\angle B \cong \angle C$, then $\angle A \cong \angle C$.

THEOREM 5-3. Congruence for triangles has the following properties:

Reflexive: $\triangle ABC \cong \triangle ABC$

Symmetric: If $\triangle ABC \cong \triangle DEF$, then $\triangle DEF \cong \triangle ABC$.

Transitive: If $\triangle ABC \cong \triangle DEF$ and $\triangle DEF \cong \triangle GHI$, then $\triangle ABC \cong \triangle GHI$.

THEOREM 5-4. (Betweenness Addition Theorem for Points) If points B and C are between A and D and $\overline{AB} \cong \overline{CD}$, then $\overline{AC} \cong \overline{BD}$.

THEOREM 5-5. (Betweenness Addition Theorem for Rays) If $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are between $\overrightarrow{OA}$ and $\overrightarrow{OD}$ and $\angle AOB \cong \angle COD$, then $\angle AOC \cong \angle BOD$.

THEOREM 5-6. If two sides of a triangle are congruent, then the angles opposite these sides are congruent.

COROLLARY 5-6-1. Every equilateral triangle is equiangular.

THEOREM 5-7. If two angles of a triangle are congruent, then the sides opposite these angles are congruent.

COROLLARY 5-7-1.	Every equiangular triangle is equilateral.
THEOREM 5-8.	The median to the base of an isosceles triangle (i) bisects the vertex angle and (ii) is perpendicular to the base.
THEOREM 5-9.	The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to it.
THEOREM 5-10.	The measure of an exterior angle of a triangle is greater than the measure of either of its non-adjacent interior angles.
THEOREM 5-11.	Given a line and a point not on the line, there is one and only one line which contains the given point and which is perpendicular to the given line.
THEOREM 6-1.	Let two distinct coplanar lines be given. If a transversal of the lines is perpendicular to each of them, then the lines are parallel.
THEOREM 6-2	Let two distinct coplanar lines be given. If two alternate interior angles determined by a transversal of the lines are congruent, then the lines are parallel.
COROLLARY 6-2-1.	Let two coplanar lines be given. If two corresponding angles determined by a transversal of the lines are congruent, then the lines are parallel.
COROLLARY 6-2-2.	Let two coplanar lines be given. If two consecutive interior angles determined by a transversal of the lines are supplementary, then the lines are parallel.
THEOREM 6-3.	Let a line and a point not on the line be given. In the plane determined by the line and the point, there is a line which contains the given point and is parallel to the given line.

THEOREM 6-4. If two distinct lines are parallel, then any two alternate interior angles determined by a transversal of the lines are congruent.

COROLLARY 6-4-1. If two distinct lines are parallel, then any two corresponding angles determined by a transversal of the lines are congruent.

COROLLARY 6-4-2. If two distinct lines are parallel, then any two consecutive interior angles determined by a transversal of the lines are supplementary.

COROLLARY 6-4-3. If a transversal is perpendicular to one of two distinct parallel lines, it is perpendicular to the other also.

THEOREM 6-5. If each of two coplanar lines is parallel to the same line, they are parallel to each other.

COROLLARY 6-5-1. If a line lies in the plane of two distinct parallel lines and intersects one of the lines in a single point, then it also intersects the other line in a single point.

COROLLARY 6-5-2. Let ξ be a plane, let S be a set of mutually parallel lines in ξ (that is, a set of lines in ξ such that each line in S is parallel to every other line in S), let T be another set of mutually parallel lines in ξ. If any one line of S is perpendicular to any one line of T, then every line of S is perpendicular to every line of T.

THEOREM 6-6. In any parallelogram, each side is congruent to the side opposite it.

THEOREM 6-7. If two sides of a quadrilateral are parallel and congruent, then the quadrilateral is a parallelogram.

THEOREM 6-8. To every pair of distinct parallel lines there is a number shich is the common length of all segments which have their respective endpoints on the given lines and are perpendicular to each of the given lines.

THEOREM 6-9. The sum of the measures of the angles of a triangle is 180 .

THEOREM 6-10. The measure of an exterior angle of a triangle is equal to the sum of the measures of its non-adjacent interior angles.

THEOREM 6-11. Given a one-to-one correspondence between the vertices of two triangles, if two pairs of corresponding angles are congruent, then the third pair of corresponding angles are congruent.

THEOREM 6-12. (The S.A.A. Theorem) Given a one-to-one correspondence between the vertices of two triangles, if two angles and a side opposite one of them in one triangle are congruent to the corresponding parts of the second triangle, the correspondence is a congruence.

THEOREM 6-13. The sum of the measures of the angles of a convex quadrilateral is 360 .

THEOREM 6-14. If one of the angles of a triangle is a right angle or an obtuse angle, then each of the other angles is an acute angle.

THEOREM 6-15. The acute angles of a right triangle are complementary.

THEOREM 6-16. (The Hypotenuse-Leg Theorem) Let a one-to-one correspondence between the vertices of two right triangles have the property that the vertices of the respective righ angles correspond. If the hypotenuse and one leg of one triangle are congruent to the corresponding parts of the other triangle, then the correspondence is a congruence.

THEOREM 6-17.	If the lengths of two sides of a triangle are unequal, then the measures of the angles opposite these sides are unequal in the same order.
THEOREM 6-18.	If the measures of two angles of a triangle are unequal, then the lengths of the sides opposite these angles are unequal in the same order.
COROLLARY 6-18-1.	The hypotenuse of a right triangle is the longest side of the triangle.
THEOREM 6-19.	The shortest segment joining a point to a line not containing the point is the segment perpendicular to the line.
COROLLARY 6-19-1.	If a line perpendicular to $\overleftrightarrow{QR}$ at the point Q contains a point P, then $PQ < PR$.
THEOREM 6-20.	If the length of one side of a triangle is equal to or greater than the length of each of the other sides, then the perpendicular segment joining the opposite vertex to this side intersects this side in an interior point of the side.
THEOREM 6-21.	(The Triangle Inequality) The sum of the lengths of any two sides of a traingle is greater than the length of the third side.
THEOREM 7-1.	The relation of similarity between convex polygons is reflexive, symmetric, and transitive.
THEOREM 7-2.	Given three coplanar parallel lines and two transversals, the corresponding segments on the transversals are proportional.
THEOREM 7-3.	If a triangle and positive number k are given, there is a triangle which is similar to the given traingle with proportionality constant k.

THEOREM 7-4. (The S.S.S. Similarity Theorem) A correspondence between two triangles such that corresponding sides are proportional is a similarity.

THEOREM 7-5. (The S.A.S. Similarity Theorem) If a correspondence between two triangles has the properties that two sides of one triangle are proportional to the corresponding sides of the other and that the included angles are congruent, then the correspondence is a similarity.

THEOREM 7-6. (The A. A. Similarity Theorem) If a correspondence between two triangles has the property that two angles of one triangle are congruent to the corresponding angles of the other, then the correspondence is a similarity.

THEOREM 7-7. In any right triangle, the altitude to the hypotenuse separates the triangle into two triangles which are similar to each other and to the original triangle.

COROLLARY 7-7-1. The square of the altitude to the hypotenuse of a right triangle is equal to the product of the projections of the legs on the hypotenuse.

COROLLARY 7-7-2. The square of the length of either leg of a right triangle is equal to the product of the lengths of the hypotenuse and the projection of that leg on the hypotenuse.

THEOREM 7-8. (The Pythagorean Theorem) In any right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the two legs.

THEOREM 7-9. (Converse of Pythagorean Theorem) If the square of the length of one side of a triangle is equal to the sum of the squares of the lengths of the other two sides, then the triangle is a right triangle with the right angle opposite the first side.

THEOREM 7-10. The triangle ABC is a right triangle with $m\angle A = 30$, $m\angle B = 60$, and $m\angle C = 90$ if and only if $(BC, CA, AB) \underset{p}{=} (1, \sqrt{3}, 2)$.

THEOREM 7-11. The triangle ABC is a right triangle with right angle at C , and with $AC = BC$, if and only if $(AC, BC, AB) \underset{p}{=} (1, 1, \sqrt{2})$.

INDEX

For precisely defined geometric terms the reference is to the formal definition. For other terms the reference is to an informal definition or to the most prominent discussion.

A.A. similarity theorem, 413
absolute value, 81 ex., 519
acute angle, 182
addition property,
  of equality, 235
  of order, 63
  of proportionality, 398
adjacent angles, 181
alternate interior angles, 319
alternation property of
  proportion, 399
altitude
  of a parallelogram, 748
  of a prism, 799
  of a pyramid, 803
  of a trapezoid, 593
  of a triangle, 427
"and", 534
angle(s), 145
  acute, 182
  adjacent, 181
  alternate interior, 319
  bisector of, 167
  central, 847
  complementary, 189
  congruent, 184
  consecutive interior, 319
  of a convex polygon, 213
  corresponding, 319
  dihedral, 215
  exterior of a polygon, 738
  exterior of a triangle, 292
  face, 789
  inscribed, 851
  intercepts an arc, 852
  interior of a polygon, 738
  interior of a triangle, 292
  linear pair of, 179
  measure of, 154
  obtuse, 182
  plane angle of a dihedral
    angle, 634
  polyhedral, 788
  of a quadrilateral, 204
  reflex, 143
  right, 182
  secant-secant, 862
  secant-tangent, 862
  side of, 145
  "straight", 144
  supplementary, 189
  tangent-chord, 861
angle(s), (con't.)
  tangent-tangent, 862
  of a triangle, 201
  trihedral, 789
  vertex of, 145
  vertical, 194
  "zero", 144
angle construction theorem, 160
antiparallel rays, 355
apothem of regular polygon, 779
arc(s)
  congruent, 855
  degree measure of, 849
  endpoints of, 848
  intercepted, 852
  length of, 900
  major, 848
  minor, 848
  of a sector, 901
  semicircle, 848
area
  of a circle, 895
  of equilateral triangle, 755
  lateral, of a prism 799
  of a parallelogram, 756
  of polygonal regions, 744
  of a rectangle, 748
  of a regular polygon, 786
  of a rhombus, 755
  of a sector of a circle, 901
  of a square, 749, 756
  of a trapezoid, 758
  of triangles, 753
area relations
  of congruent triangles, 747
  of parallelograms, 769
  of triangles, 767
A.S.A. postulate. 252
axioms, 10
base(s)
  of isosceles triangle, 277
  of a parallelogram, 748
  of a prism, 796
  of a pyramid, 803
  of a trapezoid, 593
base angles
  of isosceles triangle, 277
  of a trapezoid, 593
betweenness
  for points, 91
  for rays, 165

betweenness-addition theorem, 240
betweenness-angles theorem, 166
betweenness-coordinates theorem, 109
betweenness-distance theorem, 117
bisection
  of an angle, 167
  of a segment, 92
boundary of a polygonal region, 732
center
  of a circle, 819
  of gravity, 771 ex.
  of a regular polygon, 779
  of a sphere, 820
central angle of a circle, 847
central triangle of a regular polygon, 779
chess, 34
chord, 821
circle(s), 819
  area of, 895
  area of sector of, 901
  center of, 819
  central angle of, 847
  chord of, 821
  circumference of, 888
  circumscribed, 905
  concentric, 820
  congruent, 822
  diameter of, 821
  exterior of, 829
  great, 821
  inscribed, 905
  interior of, 829
  major arc of, 848
  minor arc of, 848
  power of a point with respect to, 871
  radius of, 819, 821
  secant of, 821
  sector of, 901
  segment of, 903
  tangent of, 830
  tangent externally, 835
  tangent internally, 835
circular-region, 894
circumference of a circle, 888
circumscribed circles, 905
circumscribed triangle, 905
collinear, 40
  in that order, 91
common external tangent, 877 ex.
common internal tangent, 877 ex.
complement, 189
components
  of directed segments, 690
  of vectors, 703
composite condition, 533
concentric, 820
conclusion, 10
concurrent lines, 597
concurrent rays, 597
  in that order, 166
concurrent segments, 597
conditionals, 244
congruence between two convex polygons, 460
congruence between two triangles, 228
congruent
  angles, 184
  arcs, 855
  chords, 834
  circles, 822
  polygons, 405
  segments, 115
  spheres, 822
  triangles, 229
consecutive interior angles, 319
constant of proportionality, 393
contrapositive, 328
  property of, 329
converse, 280
  of Pythagorean theorem, 434
convex polygon(s), 211
  angles of, 213
  consecutive angles of, 213
  diagonals of, 212
  interior of, 212
convex polyhedron, 884
convex set of points, 134
coordinate planes, 643
coordinate of a point, 76
coordinate system, 76
  in a plane, 509
  in space, 641
  on a line, 505
  origin of, 76
  unit point of, 76
coordinates of a point
  in a plane, 511
  in space, 646
coplanar, 44
correspondence, one-to-one, 30
  between triangles, 227
corresponding angles, 319
counter-example, 5
counting numbers, 55
cross-section of a prism, 798
cube, 797
decagon, 210

decahedron, 784
deductive reasoning, 10
definitions, 15
    circular, 34
    complete form, 241
    formal, 15
    if and only if form, 242
    in proofs, 241
degree, 154
degree measure of an arc, 849
diagonals of a convex polygon, 212
diameter, 821
dihedral angle(s), 215
    edge of, 215
    face of, 215
    measure of, 635
    plane angle of, 634
    right, 635
    vertical, 216
directed segment(s), 684
    equal, 685
    equivalent, 686
        properties of, 687
    length of, 685
    opposite of, 693
    product with a number, 693
    subtraction of, 701 ex.
    sum of, 697
    x-component of, 690
    y-component of, 690
displacement, 683
distance,
    between a point and a line, 376
    between a point and a plane, 628
    between two parallel lines, 354
    between two points, 522, 655
    measure of, 70
distance formula,
    in a plane, 522
    in space, 655
dodecagon, 210
dodecahedron, 784
"dot product", 718
edge of
    halfplane, 138
    polygonal-region, 734 ex.
    polyhedral angle, 788
    polyhedron, 783
empty set, 26
endpoints of an arc, 848
equal directed segments, 685
equal vectors, 704
equation(s), 538
    equivalent, 539
    intercept form, 571 ex.
    parametric, 546, 658
    of a plane, 663
    point-slope form, 569
    slope-intercept form, 571 ex.
    two-point form, 569
equiangular triangle, 277
equilateral triangle, 277
    area of, 755
equivalent directed segments, 686
equivalent equations, 539
Euler's theorem, 734 ex.
exterior
    of an angle, 176
    of a circle, 829
    of a sphere, 841
    of a triangle, 203
exterior angle of polygon, 738
exterior angle of triangle, 292
external secant segment, 870
externally tangent circles, 835
face angle of a polydehral angle, 789
faces
    of a polygonal-region, 734 ex.
    of a polyhedral angle, 789
    of a polyhedron, 783
foot of a perpendicular, 294
frustum of a pyramid, 805
geometrical applications of vectors, 714
grad, 153
graph, 514
great circle of a sphere, 821
greater than, 57
halfline, 137
halfplane, 138
    edge of, 138
halfspace, 139
heptagon, 210
heptahedron, 784
hexagon, 210
hexahedron, 784
horizontal lines, 510
hypotenuse, 366
hypotenuse-leg theorem, 367
hypothesis, 10
icosahedron, 784
identity correspondence, 33 ex.
if and only if form, 242
if-then form, 17
incidence relations, 35
    points and lines, 36
    points, lines, and planes, 42

indirect method of proof, 325
indirect reasoning, 12
inductive reasoning, 5
inequalities, 538
  in the same order, 370
initial point, 684
inscribed angle, 851
inscribed circle, 905
inscribed triangle, 905
integers, 56
  negative, 56
  positive, 55
intercept form of a linear
  equation, 571
intercepted arc, 852
interior
  of an angle, 175
  of a circle, 829
  of a convex polygon, 212
  of a polygonal-region, 732
  of a ray, 90
  of a segment, 90
  of a sphere, 841
  of a triangle, 202
interior angle
  of a polygon, 738
  of a triangle, 292
internally tangent circles, 835
intersect, 27
intersection of sets, 24, 534
inversely proportional, 766
inversion property of
  proportion, 399
isosceles trapezoid, 593
isosceles triangle, 277
  base of, 277
  base angles of, 277
  theorem, 275
  vertex of, 277
lateral area of a prism, 799
lateral edge of a prism, 797
lateral face of a prism, 797
lateral surface of a prism, 797
leg of a right triangle, 366
leg of a trapezoid, 593
length of an arc, 900
length of a segment, 114
length of a vector, 704
less than, 64
limit, 888
line(s)
  concurrent, 597
  coordinate system on, 505
  horizontal, 510
  intercept form of, 571 ex.
  opposite sides of, 138
line(s), (con't.)
  parallel, 316, 343
  parallel to a plane, 617
  parametric equations of, 546
  perpendicular, 183
  perpendicular to a plane, 610
  point-slope form of, 569
  projection of a point on, 428
  projection of a
    segment on, 428
  representation of, 35
  skew, 316
  slope of, 556
  slope-intercept
    form of, 571 ex.
  transversal, 317
  two-point form of, 569
  undefined, 33
  vertical, 510
linear pair, 179
Lobachevskian geometry, 340
locus, 538
logical equivalence, 329
logical system, 2
magnitude of a vector, 704
major arc, 848
measure of an angle, 154
measure of an arc, 849
measure of a dihedral angle, 635
measure of distance, 70
median of a trapezoid, 593, 758
median of a triangle, 289
midpoint of a
  segment, 91, 526, 550
midray, 167
mil, 153
minor arc, 848
multiplication property
  of equality, 236
  of order, 63
nonagon, 210
nonahedron, 784
non-Euclidean geometries, 339
null set, 29 ex.
numbers
  counting, 55
  inequality of, 57
  integers, 56
  irrational, 56
  natural, 55
  negative, 63
  order properties, 63
  positive, 63
  rational 56
  real, 56
obtuse angle, 182
octagon, 210
octahedron, 784

one-to-one correspondence, 30
  between triangles, 228
"or", 534
order,
  for real numbers, 63
  of collinear points, 91
ordered pair, 511
ordered triple, 646
origin, 76, 510, 684
origin and unit point theorem, 78
outer end of radius, 821
parallel lines, 316, 343
  distance between, 354
  properties of, 347
parallel postulate, 339
parallel rays, 355
parallel segments, 350
parallel vectors, 705
parallelepiped, 797
  rectangular, 797
parallelism, 315
  of a line to a plane, 617
  of two planes, 617
parallelogram, 351
  altitude of, 748
  area of 756
  base of, 748
  properties of, 603
parameter, 546
parametric equations
  in a plane, 546
  in space, 658
pentagon, 210
pentahedron, 784
perpendicular, 183
  foot of, 294
  lines, 183
  planes, 635
  sets, 183
  vectors, 717
perpendicularity of a line and a plane, 610
pi, π , 889
plane(s),
  coordinate system in, 509
  equation of, 663
  parallel to a line, 617
  parallel to another plane, 617
  perpendicular, 635
  perpendicular to a line, 610
  representation of, 44
  tangent, 842
  undefined, 33
plane angle of a dihedral angle, 634
plane separation postulate, 138
plot, 514
point(s)
  line coordinate of, 76
  plane coordinates of, 511
  space coordinates of, 646
  distance between, 522, 655
  power of with respect to a circle, 871
  representation of, 35
  undefined, 33
point plotting theorem, 116
point-slope form of a linear equation, 569
point of tangency
  of a circle, 830
  of a sphere, 842
polygon(s)
  angles of, 213
  congruence between, 405
  consecutive sides of, 210
  consecutive verices of, 210
  convex, 211
  regular, 297, 779
  sides of, 209
  similar, 403
  vertex of, 209
polygonal-region(s), 730
  area of, 744
  boundary of, 732
  edges of, 734
  faces of, 734
  interior of, 732
  vertices of, 734
polyhedral angle(s), 788
  edge of, 788
  face of, 789
  face angle of, 789
  vertex of, 788
polyhedron(s), 783
  convex, 784
  edge of, 783
  face of, 783
  regular, 784
  section of, 784
  vertex of, 783
postulate(s), 10
  of Algebra, 57
  of congruence,
    A.S.A., 252
    S.A.S., 251
    S.S.S., 253
  of incidence, 35, 36, 42
  interior of an angle, 174
  parallel, 339
  plane separation, 138
  proportional segments, 412
  protractor, 159
  ruler, 77

- power of a point with respect to a circle, 871
- prime number, 5
- prism(s), 796
  - altitude of, 799
  - base of, 796
  - cross-section of, 798
  - lateral area of, 799
  - lateral edge of, 797
  - lateral face of, 797
  - lateral surface of, 797
  - rectangular, 796
  - right, 797
  - right-section of, 798
  - total area of, 799
  - triangular, 796
- prismatic surface, 797
- product property of proportion, 399
- projection,
  - of a point into a plane, 630
  - of a point on a line, 428
  - of a segment on a line, 428
  - of a set of points into a plane, 631
  - of a vector, 720
- proof, 17
  - finding of, 271
  - indirect method, 325
  - paragraph form, 276
  - two column form, 244, 276
  - using definitions in, 241
  - writing of, 260
- properties of
  - congruence, 233
    - for angles, 235
    - for segments, 234
    - for triangles, 235
  - directed segments, 687
  - equality, 233, 235
  - order, 63
  - parallel lines, 347
  - parallel planes, 627
  - parallelograms, 603
  - proportion, 399
  - proportionality, 397
  - rectangles, 603
  - rhombuses, 603
  - scalar products, 719
  - similar convex polygons, 406
  - squares, 603
  - trapezoids, 603
  - vectors, 707
- property of the contrapositive, 329
- proportion, 399
  - properties of, 399
- proportional, 393
- proportional segments postulate, 412
- proportionality, 392
  - inverse, 766
  - properties of, 397
- protractor, 151
- protractor postulate, 159
- pyramid(s), 803
  - altitude of, 803
  - base of, 803
  - frustum of, 805
  - regular, 804
  - slant height of, 805
  - vertex of, 803
- Pythagorean theorem, 433
- quadrants, 513
- quadrilateral(s), 204
  - opposite sides of, 213
  - opposite vertices of, 213
  - sides of, 204
  - vertices of, 204
- radian, 153
- radius
  - of a circle, 819, 821
  - outer end of, 821,
  - of a regular polygon, 779
  - of a sector of a circle, 901
- ray(s), 84
  - antiparallel, 355
  - concurrent, 597
  - coordinate of, 159
  - endpoint of, 84
  - initial, 143
  - interior of, 90
  - opposite, 85
  - ordered pair of, 143
  - parallel, 355
  - slope of, 556
  - terminal, 143
- ray-coordinate system, 159
- real numbers, 56
- reasoning,
  - deductive, 10
  - indirect, 12
  - inductive, 5
- rectangle, 578
  - area of, 748
  - properties of, 603
- rectangular parallelepiped, 797
- rectangular prism, 796
- reflex angle, 143
- reflexive property
  - of congruence,
    - for angles, 235
    - for segments, 234
    - for triangles, 235
  - of equality, 233
  - of equivalent directed segments, 687

reflexive property, (con't.)
   of parallel lines, 347
   of parallel planes, 627
   of proportionality, 397
   of similar convex
      polygons, 406
regular polygon(s), 297
   apothem of, 779
   area of, 780
   center of, 779
   central triangle of, 779
   radius of, 779
regular polyhedron, 784
regular pyramid, 804
resultant, 709 ex.
rhombus, 578
   area of, 755
   properties of, 603
Riemannian geometry, 340
right angle, 182
right dihedral angle, 635
right prism, 797
right section of a prism, 798
right triangle, 366
rotation, 143
ruler postulate, 77
S.A.A. theorem, 361
S.A.S. postulate, 251
S.A.S. similarity theorem, 421
scalar(s), 683, 703
scalar product, 718
   properties of, 719
secant, 821
secant-secant angle, 862
secant-segment, 870
   external, 870
section of polyhedron, 784
sector of a circle, 901
   arc of, 901
   area of, 901
   radius of, 901
segment(s), 86
   of a circle, 903 ex.
   concurrent, 597
   congruent, 115
   directed, 684
   endpoints of, 86
   interior of, 90
   length of, 114
   midpoint of, 91, 526, 550
   parallel, 350
   slope of, 554
   tangent, 869
semicircle, 848
separation,
   by a line, 138
   by a plane, 139
   by a point, 133
set(s), 19
   convex, 134
   elements of, 19
   empty, 26
   equality of, 20
   intersection of, 24, 534
   null, 29 ex.
   of real numbers, 55
   union of, 25, 534
set-builder notation, 531
side,
   of an angle, 145
   of a line, 138
   of a plane, 139
   of a quadrilateral, 204
   of a triangle, 201
similar polygons, 403
   properties of, 406
skew lines, 316
slant height of a pyramid, 804
slope,
   of a segment, 554
   of a non-vertical line, 556
   of a non-vertical ray, 556
slope-intercept form of a
   linear equation, 571
space, 36
   coordinate system in, 641
sphere(s), 820
   center of, 820
   chord of, 821
   concentric, 820
   congruent, 822
   diameter of, 821
   exterior of, 841
   great circle of, 821
   interior of, 841
   radius of, 820, 821
   secant of, 821
   tangent to, 842
square, 578
   area of, 749, 756
   properties of, 603
S.S.S. postulate, 253
S.S.S. similarity theorem, 420
"straight angle", 144
subset, 22
   proper, 25 ex.
substitution property, 233
supplement, 189
supplement theorem, 189
symmetric property
   of congruence,
      for angles, 235
      for segments, 234
      for triangles, 235
   of equality, 233
   of equivalent directed
      segments, 687

- symmetric property, (con't.)
  - of parallel lines, 347
  - of parallel planes, 627
  - of proportionality, 397
  - of similar convex polygons, 406
- tangent to a circle, 830
  - common external, 877 ex.
  - common internal, 877 ex.
- tangent-chord angle, 861
- tangent circles, 835
- tangent plane, 842
- tangent-secant angle, 862
- tangent-segment, 869
- tangent-tangent angle, 862
- terminal point, 684
- terminus, 684
- tetrahedron, 784
- theorem(s), 10
  - A.A. similarity, 422
  - angle construction, 160
  - betweenness-angle, 166
  - betweenness-addition,
    - for points, 240
    - for rays, 240
  - betweenness-coordinate, 109
  - betweenness-distance, 117
  - hypotenuse-leg, 367
  - isoceles triangle, 275
  - origin and unit point, 78
  - point-plotting, 116
  - Pythagorean, 433
    - converse of, 434
  - S.A.A., 361
  - S.A.S. similarity, 421
  - S.S.S. similarity, 420
  - supplement, 189
  - triangle inequality, 378
  - two-coordinate system, 103
  - two-point, 108
- total area of a prism, 799
- transitive property
  - of congruence,
    - for angles, 235
    - for segments, 234
    - for triangles, 235
  - of equality, 234
  - of equivalent directed segments, 687
  - of order, 63
  - of parallel lines, 347
  - of parallel planes, 627
  - of proportionality, 398
  - of similar convex polygons, 406
- transversal, 317
- trapezoid, 593
  - altitude of, 593
  - area of, 758
  - base of, 593
  - base angles of, 593
  - isoceles, 593
  - legs of, 593
  - median of, 593, 758
  - properties of, 603
- triangle(s), 201
  - altitude of, 427
  - angles of, 201
  - area of, 753
  - circumscribed, 905
  - congruent, 229
  - equiangular, 277
  - equilateral, 277
  - exterior of, 203
  - exterior angle of, 292
  - inscribed, 905
  - interior of, 202
  - interior angle of, 292
  - isosceles, 277
  - median of, 289
  - right, 366
  - sides of, 201
  - vertices of, 201
- triangle inequality theorem, 378
- triangular prism, 796
- triangular-region, 730
- trihedral angle, 789
- two-coordinate system theorem, 103
- two-point form of a linear equation, 569
- two-point theorem, 108
- unequal in the same order, 370
- union of sets, 25, 534
- unit, 70
- unit area, 744
- unit-pair, 70
- unit-point, 76
- unit square, 748
- vector(s), 700, 703
  - components of, 703
  - equal, 704
  - geometrical applications, 714
  - length of, 704
  - magnitude of, 704
  - parallel, 705
  - perpendicular, 717
  - product with scalar, 704
  - projection of, 718
  - properties of, 707
  - scalar product of, 718
  - subtraction of, 705
  - sum of, 705
  - zero, 704

vertex
   of a polygonal-region, 734 ex.
   of a polyhedral angle, 788
   of a polyhedron, 783
   of a pyramid, 803
vertex angle of an isosceles triangle, 277
vertical angles, 194
vertical lines, 510
x-axis, 510
x-component of a directed segment, 690
x-coordinate, 511, 646
xy-plane, 510, 643
xz-plane, 643
y-axis, 510
y-component of a directed segment, 690
y-coordinate, 511, 646
yz-plane, 643
z-coordinate, 646
zero-ray, 159
zero vector, 704